

Ising vectors and automorphism groups of commutant subalgebras related to root systems

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Abstract In this article we study and obtain a classification of Ising vectors in vertex operator algebras (VOAs) associated to binary codes and $\sqrt{2}$ times root lattices, where an Ising vector is a conformal vector with central charge $1/2$ generating a simple Virasoro sub-VOA. Then we apply our results to study certain commutant subalgebras related to root systems. We completely classify all Ising vectors in such commutant subalgebras and determine their full automorphism groups.

1 Introduction

Motivated by the problem of looking for maximal associative subalgebras of the monstrous Griess algebra [12], a class of conformal vectors in the lattice vertex operator algebra (VOA) $V_{\sqrt{2}R}$ were studied and constructed in [7], where R is a root lattice of type A, D or E of rank ℓ and $\sqrt{2}R$ denotes $\sqrt{2}$ times an ordinary root lattice R . We adopt the standard notation for lattice VOAs as in [10]. In [7], Dong et al. constructed conformal vectors of $V_{\sqrt{2}R}^+$ of the forms

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$$s_R = \frac{h}{h+2}\omega - \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha} \quad \text{and} \quad \tilde{\omega}_R = \frac{2}{h+2}\omega + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha},$$

where ω is the Virasoro element of $V_{\sqrt{2}R}^+$, h is the Coxeter number of R and $\Phi(R)$ denotes the root system of R . The central charges of s_R and $\tilde{\omega}_R$ are $\ell h/(h+2)$ and $2\ell/(h+2)$, respectively. The Weyl group $W(R)$ of the root system $\Phi(R)$ induces a natural action on the lattice vertex operator algebra $V_{\sqrt{2}R}$ and its $\mathbb{Z}_2$ -orbifold $V_{\sqrt{2}R}^+$. By the construction, both conformal vectors s_R and $\tilde{\omega}_R$ are fixed by $W(R)$ so that $W(R)$ acts identically on the Virasoro vertex operator subalgebra $\text{Vir}(\tilde{\omega}_R)$ generated by $\tilde{\omega}_R$. Therefore, the commutant (or coset) subalgebra

$$\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R)) = \left\{ a \in V_{\sqrt{2}R}^+ \mid [Y(a, z_1), Y(u, z_2)] = 0 \quad \text{for all } u \in \text{Vir}(\tilde{\omega}_R) \right\}$$

affords an action of the Weyl group $W(R)$.

In this article, we shall study the structure of the commutant vertex operator subalgebra $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ of $V_{\sqrt{2}R}^+$. As our main result, we shall determine all the conformal vectors of central charge $1/2$ in $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ and show that the VOA $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ is generated by its weight two subspace. We shall also determine the full automorphism group of $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ and show that it always contains a half of the Weyl group $W(R)$.

Besides its own interest, the study of $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ also has a perspective on the Monster simple group. There is an attempt in [22, 23] to elucidate McKay's observation on the Monster simple group and the extended E_8 diagram via VOAs. It is known (cf. [3, 28]) that a $2A$ -involution of the Monster is in one-to-one correspondence with a conformal vector with central charge $1/2$ in the moonshine VOA [10] via so-called Miyamoto involution. Taking notice of a fact that the conformal vector $\tilde{\omega}_{E_8}$ of $V_{\sqrt{2}E_8}^+$ is of central charge $1/2$, Lam et al. used $V_{\sqrt{2}E_8}$ to relate the E_8 diagram with the Monster in [22, 23]. To each node of the extended E_8 diagram, they constructed two conformal vectors of central charge $1/2$ inside $V_{\sqrt{2}E_8}$ which shall induce two $2A$ -involutions of the Monster [22]. Then they used commutant subalgebra $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(s_R))$ for a sublattice R of E_8 to study the subalgebra generated by two such conformal vectors of $V_{\sqrt{2}E_8}$ in [23]. Since s_R and $\tilde{\omega}_R$ are mutually commutative conformal elements in $V_{\sqrt{2}R}^+$ and the sum $s_R + \tilde{\omega}_R$ is the Virasoro vector of $V_{\sqrt{2}R}^+$, the subalgebra $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ is exactly equal to the commutant subalgebra of $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(s_R))$ in $V_{\sqrt{2}R}^+$ (cf. [11]). By this duality, we can relate the structure of $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ with the centralizer of two $2A$ -involutions of the Monster. In [14], Glauberman and Norton suggested some interesting relations between Weyl groups and centralizers of two $2A$ -involutions of Monster. Since $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ naturally affords an action of the Weyl group $W(R)$, the study of $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ may lead Glauberman–Norton's observation to an appreciable settlement.

Let us denote $\text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega}_R))$ by M_R for simplicity of notation. The structure of M_R is closely related to that of the root system $\Phi(R)$ of R , so the automorphism group of M_R has a similar structure to the Weyl group $W(R)$ of R . The Weyl group $W(R)$ is a 3-transposition group, and Miyamoto discovered in [28] that the 3-transposition

property of $W(R)$ acting on $V_{\sqrt{2}R}^+$ comes from the structure of conformal vectors with central charge $1/2$. In [28], Miyamoto introduce a way to define involutions of vertex operator algebras containing conformal vectors with central charge $1/2$. He showed that in some cases these involutions generate a 3-transposition group, which is exactly the case for $V_{\sqrt{2}R}^+$. There are many results about the group generated by these involutions, see [16–18, 22, 23, 25, 28, 36]. Recently, Matsuo classified all 3-transposition groups defined by conformal vectors with central charge $1/2$ in [25]. According to his classification, the parameters of M_R such as central charges or dimensions of weight two subspace coincide with those for 3-transposition groups in his list. In particular, if we take $R = E_6, E_7$, or E_8 , then the corresponding groups have nice symmetry. In fact, this is one of the main motivations of our work. We shall determine the automorphism group of M_R by studying conformal vectors of central charge $1/2$ inside M_R .

Let us explain our result more precisely. Since the conformal vector with central charge $1/2$ plays a central role in our discussion, we will refer a conformal vector with central charge $1/2$ to as an *Ising vector* if it generates a simple Virasoro vertex operator subalgebra. To study Ising vectors, we first study a class of vertex operator algebras, called *code VOAs* (cf. [29]). In the case that a code VOA has finitely many Ising vectors, we classify all the Ising vectors of a code VOA. Moreover, we will present a formula to count the exact number of such conformal vectors in Corollary 3.9. Then for a (not necessarily indecomposable) root lattice R we will classify all Ising vectors inside $V_{\sqrt{2}R}^+$ by giving an embedding of $V_{\sqrt{2}R}^+$ into a code VOA. As a result, it is shown in Theorem 4.6 that for each Ising vector e of $V_{\sqrt{2}R}^+$, there exists a sublattice K of $\sqrt{2}R$ isometric to either $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_K^+ \subset V_{\sqrt{2}R}^+$. We expect that this is true not only for a lattice VOA associated to $\sqrt{2}R$ but also for any lattice without roots (cf. Remark 4.7). The classification of Ising vectors of M_R immediately follows from that of $V_{\sqrt{2}R}^+$. We also show that M_R is generated by its Ising vectors. This reduces the study of $\text{Aut}(M_R)$ to an analysis of the permutation group of the Ising vectors. Since we know the group generated by involutions associated to these Ising vectors by the results obtained in [16, 25], it is not difficult to determine the permutation group of the Ising vectors.

The organization of this paper is as follows. In Sect. 2 we prepare some basic notation and related facts about conformal vectors and commutant subalgebras. In Sect. 3 we study Ising vectors of code VOAs. We present a classification of Ising vectors of a code VOAs in Proposition 3.8. In Sect. 4 we study Ising vectors of lattice VOAs. We classify Ising vectors of $V_{\sqrt{2}R}^+$ and M_R with R an irreducible root lattice. In Sect. 5 we determine the shape of $\text{Aut}(M_R)$. We make use of the fact that M_R is generated by its Ising vectors. In the case that $R = E_6$ or E_7 , the proof of this fact is rather technical and long. So we separately give it in Sect. 7. In Sect. 6 we study a relation between inductive structures of a 3-transposition group acting on a VOA and its commutant subalgebras. As an example, we study a VOA corresponding to an inductive structure $O_{10}^+(2)^{(2)} \simeq O_8^-(2)$. Section 7 is devoted to the proof that M_R with $R = E_6$ or E_7 is generated by its weight two subspace as a VOA. We use some representation theory of the unitary Virasoro VOAs and W -algebras there. So we also review some facts about these VOAs.

1.1 Notation

In this article, every VOA we considered is defined over $\mathbb{C}$ and supposed to have the grading $V = \bigoplus_{n \geq 0} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$. For a VOA structure $(V, Y(\cdot, z), \mathbf{1}, \omega)$ on V , the vector ω is called the *Virasoro element* or *Virasoro vector* of V . For simplicity, we often use (V, ω) to denote the structure $(V, Y(\cdot, z), \mathbf{1}, \omega)$. The vertex operator $Y(a, z)$ of $a \in V$ is expanded such as $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$.

An element $u \in V$ is referred to as a *conformal vector* with central charge $c_u \in \mathbb{C}$ if $u \in V_2$ and it satisfies $u_{(1)}u = 2u$ and $u_{(3)}u = c_u\mathbf{1}$. It is well known (cf. [17, 28]) that after setting $L^u(n) := u_{(n+1)}, n \in \mathbb{Z}$, we obtain a representation of the Virasoro algebra on V :

$$[L^u(m), L^u(n)] = (m - n)L^u(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c_u.$$

For $c, h \in \mathbb{C}$, we denote by $L(c, h)$ the irreducible highest weight module over the Virasoro algebra with central charge c and highest weight h . It is well known that $L(c, 0)$ has a simple VOA structure (cf. [11]).

For a positive definite even lattice L , we will denote the lattice VOA associated to L by V_L (cf. [10]). We adopt the standard notation for V_L as in [10]. In particular, V_L^+ denotes the fixed point subalgebra of V_L under the lift of (-1) -isometry on L .

Given an automorphism group G of V , we denote by V^G the G -fixed point subalgebra of V . The subalgebra V^G is called the G -orbifold of V in the literature.

We denote the ring $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$. Let $C \subset \mathbb{Z}_2^n$ be a linear code. For a codeword $\alpha = (\alpha_1, \dots, \alpha_n) \in C$, we define the support of α by $\text{supp}(\alpha) := \{i \mid \alpha_i = 1\}$. For a subset A of C , we define $\text{supp}(A) := \bigcup_{\alpha \in A} \text{supp}(\alpha)$.

2 Commutant subalgebras and conformal vectors

We will present the notion of a commutant subalgebra and its description by a pair of mutually commutative conformal vectors.

2.1 Commutant subalgebras

Let (V, ω) be a VOA. A decomposition $\omega = u^1 + \dots + u^n$ is called *orthogonal* if u^i are conformal vectors and they are mutually commutative, i.e., $[Y(u^i, z_1), Y(u^j, z_2)] = 0$ if $i \neq j$. The following lemma is well known (cf. Theorem 5.1 of [11]).

Lemma 2.1 *Assume that a VOA (V, ω) has a grading $V = \bigoplus_{n \geq 0} V_n$ such that $V_0 = \mathbb{C}\mathbf{1}$ and $V_1 = 0$. Then for any conformal vector u of V , the decomposition $\omega = u + (\omega - u)$ is orthogonal.*

In this paper, a sub-VOA of V is defined by means of a pair (U, e) of a subalgebra U of V containing the vacuum element $\mathbf{1}$ of V and a conformal vector e of $V_2 \cap U$ such that e is the Virasoro vector of U and the structure (U, e) inherits the grading of V , that is, $U = \bigoplus_{n \geq 0} U_n$ with $U_n = V_n \cap U$. In the case that the conformal vector of a sub-VOA of V coincides with that of V , we will refer such a sub-VOA to as a *full* sub-VOA. Every conformal vector u of V together with the vacuum $\mathbf{1}$ generates a Virasoro sub-VOA of V which we will denote by $\text{Vir}(u)$.

Let S be a subset of V . The following is easy to see.

Lemma 2.2 *The subspace $S^c := \{a \in V \mid a_{(i)}S = 0, i \geq 0\}$ forms a subalgebra and satisfies that $[Y(u, z_1), Y(v, z_2)] = 0$ for any $u \in S, v \in S^c$.*

By the lemma above, we define the *commutant subalgebra* of a subalgebra U of V by

$$\text{Com}_V(U) := U^c = \{a \in V \mid a_{(i)}U = 0, i \geq 0\}. \quad (2.1)$$

If U has a conformal vector e such that (U, e) forms a sub-VOA of V , then it is shown in Theorem 5.2 of [11] that $\text{Com}_V(U) = \ker_V e_{(0)}$. Therefore, the commutant subalgebra is described in term of the conformal vector e of U .

2.2 Conformal vectors associated to root systems

Let L be an integral lattice. A vector $\alpha \in L$ is called a *root* of L if $\langle \alpha, \alpha \rangle = 2$, and L is said to be a *root lattice* if it is generated by its roots. We often denote the set of roots of L by $\Phi(L)$ and call it the *root system* of L . In this article, we only consider simply laced root systems. Let R be a root lattice with root system $\Phi(R)$ of type ADE.

Denote by ℓ the rank of R and by h the Coxeter number of R . We denote by $\sqrt{2}R$ the lattice whose norm is twice of R 's. We consider the fixed point subalgebra $V_{\sqrt{2}R}^+$ of the lattice VOA $V_{\sqrt{2}R}$ under the lift of (-1) -isometry on R . It is clear that $V_{\sqrt{2}R}^+$ has a grading $V_{\sqrt{2}R}^+ = \bigoplus_{n \geq 0} (V_{\sqrt{2}R}^+)_n$ such that $(V_{\sqrt{2}R}^+)_0 = \mathbb{C}\mathbf{1}$ and $(V_{\sqrt{2}R}^+)_1 = 0$. We can find conformal vectors of $V_{\sqrt{2}R}^+$ defined as follows. Set

$$s = s_R := \frac{h}{h+2}\omega - \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}R}^+, \quad (2.2)$$

where ω is the Virasoro vector of $V_{\sqrt{2}R}^+$. Then it is shown in [7] that s defines a conformal vector with central charge $\ell h/(h+2)$. By Lemma 2.1,

$$\tilde{\omega} = \tilde{\omega}_R := \omega - s = \frac{2}{h+2}\omega + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}R}^+ \quad (2.3)$$

is also a conformal vector with central charge $2\ell/(h+2)$ and the decomposition $\omega = s + \tilde{\omega}$ is orthogonal. In this article, we will mainly consider the commutant subalgebra

$$M_R := \text{Com}_{V_{\sqrt{2}R}^+}(\text{Vir}(\tilde{\omega})) = \ker_{V_{\sqrt{2}R}^+} \tilde{\omega}_{(0)}. \quad (2.4)$$

It is clear from the expression in (2.3) that $\tilde{\omega}$ is invariant under the natural action of the Weyl group $W(R)$ associated to the root system $\Phi(R)$. Therefore, the commutant subalgebra M_R naturally affords an action of the Weyl group $W(R)$.

3 Ising vectors and Ising frames

We will introduce the notion of an Ising vector by which one can define an involution of a VOA. We will review basic facts about involutions associated to Ising vectors. Then we will study the classification of Ising vectors of a code vertex operator algebra. We will also study automorphisms of a code vertex operator algebra. A brief description of the automorphism group of a code VOA will be presented.

3.1 Miyamoto involution

We begin by the definition of an Ising vector.

Definition 3.1 A conformal vector e of a VOA V is called an Ising vector if the subalgebra $\text{Vir}(e)$ generated by e is isomorphic to the simple Virasoro VOA $L(1/2, 0)$ with central charge $1/2$. An orthogonal decomposition $\omega = e^1 + \cdots + e^n$ of the Virasoro vector ω is called an Ising frame if each e^i is an Ising vector.

Remark 3.2 An Ising vector is often referred to as a rational conformal vector of central charge $1/2$ in the literature (cf. [17, 28]).

It is well known that the Virasoro VOA $L(1/2, 0)$ is rational and has three irreducible representations, $L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$ (cf. [9]).

Let e be an Ising vector of a VOA V . Since $\text{Vir}(e)$ is rational, V is a semisimple $\text{Vir}(e)$ -module. For $h = 0, 1/2, 1/16$, denote by $V_e(h)$ the sum of all irreducible $\text{Vir}(e)$ -submodules of V isomorphic to $L(1/2, h)$. Then we have the isotypical decomposition:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16). \quad (3.1)$$

Define a linear automorphism τ_e on V which acts on $V_e(0) \oplus V_e(1/2)$ by identity and on $V_e(1/16)$ by -1 . Then it is shown in [28] that $\tau_e \in \text{Aut}(V)$. On the $\langle \tau_e \rangle$ -fixed point subalgebra $V^{(\tau_e)} = V_e(0) \oplus V_e(1/2)$, define a linear automorphism σ_e which acts on $V_e(0)$ by identity and on $V_e(1/2)$ by -1 . Then it is also shown in [28] that $\sigma_e \in \text{Aut}(V^{(\tau_e)})$. We will refer $\tau_e \in \text{Aut}(V)$ (resp. $\sigma_e \in \text{Aut}(V^{(\tau_e)})$) to as the *Miyamoto involution* of τ -type (resp. σ -type). An Ising vector e of V is called of σ -type if τ_e defines an identity automorphism on V , and we also refer an Ising frame $\omega = e^1 + \cdots + e^n$ of V to as of σ -type if all e^i , $1 \leq i \leq n$, are of σ -type on V .

3.2 Code VOA

Let us review the construction of code VOAs in [29] at least for what we need in this paper.

Let $\mathcal{A}$ be the algebra generated by $\{\psi_r \mid r \in \mathbb{Z} + 1/2\}$ subject to the defining relation $\psi_r \psi_s + \psi_s \psi_r = \delta_{r+s, 0}$, $r, s \in \mathbb{Z} + 1/2$. Let $\mathcal{A}^+$ be the subalgebra of $\mathcal{A}$ generated by $\{\psi_r \mid r > 0\}$ and let $\mathbb{C}\mathbf{1}$ be a trivial $\mathcal{A}$ -module. Then set $X := \text{Ind}_{\mathcal{A}^+}^{\mathcal{A}} \mathbb{C}\mathbf{1}$. Consider the generating function

$$\psi(z) := \sum_{n \in \mathbb{Z}} \psi_{n+1/2} z^{-n-1}. \quad (3.2)$$

It is well known that the space X , with the standard $\mathbb{Z}_2$ -grading, has a unique structure of a simple vertex operator superalgebra (SVOA for short) with the vacuum element $\mathbf{1}$ such that $Y(\psi_{-1/2} \mathbf{1}, z) = \psi(z)$. The vector $\omega = \frac{1}{2} \psi_{-3/2} \psi_{-1/2} \mathbf{1}$ is a Virasoro vector of X with central charge $1/2$ and the quadruple $(X, Y(\cdot, z), \mathbf{1}, \omega)$ is isomorphic to $L(1/2, 0) \oplus L(1/2, 1/2)$ (cf. [15]).

Set $X^0 := L(1/2, 0) \subset X$ and $X^1 := L(1/2, 1/2) \subset X$ under the isomorphism $X \simeq L(1/2, 0) \oplus L(1/2, 1/2)$. Then $X^{\otimes n}$ also forms an SVOA as a tensor product of SVOAs. For an even linear subcode C of $\mathbb{Z}_2^n$, set

$$V_C := \bigoplus_{\alpha=(\alpha_1, \dots, \alpha_n) \in C} X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n}, \quad (3.3)$$

which is a subalgebra of $X^{\otimes n}$. This is a simple VOA called a *code VOA* associated to C (cf. [29]). A code VOA has the standard Ising frame of σ -type and the following theorem characterizes code VOAs via their standard Ising frames.

Theorem 3.3 ([30]) *Let V be a simple VOA with an Ising frame $\omega = e^1 + \cdots + e^n$ of σ -type. Then there exists a unique even linear subcode C of $\mathbb{Z}_2^n$ such that V is isomorphic to a code VOA V_C with respect to the Ising frame $\omega = e^1 + \cdots + e^n$.*

We will use the following notation for code VOAs. Set $u^0 := \mathbf{1} \in X^0$, $u^1 := \psi_{-1/2}\mathbf{1} \in X^1$, and for a codeword $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^n$, we set

$$x^\alpha := u^{\alpha_1} \otimes \cdots \otimes u^{\alpha_n} \in X^{\otimes n}. \quad (3.4)$$

Then x^α is a highest weight vector of $X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n}$ with norm $\langle x^\alpha, x^\alpha \rangle = \pm 1$,¹ where the invariant bilinear form $\langle \cdot, \cdot \rangle$ on X is normalized such as $\langle \mathbf{1}, \mathbf{1} \rangle = 1$.

For a codeword $\beta \in \mathbb{Z}_2^n$, the subspace

$$V_{C+\beta} := \bigoplus_{\gamma=(\gamma_1, \dots, \gamma_n) \in C+\beta} X^{\gamma_1} \otimes \cdots \otimes X^{\gamma_n}$$

uniquely forms an irreducible V_C -submodule of $X^{\otimes n}$ (cf. [30]). We will call $V_{C+\beta}$ a *coset type module* over V_C .

3.3 Ising vectors of σ -type

In this subsection we consider a VOA V with trivial weight one subspace, i.e., $V_1 = 0$. Then the weight two subspace V_2 equipped with the product $a \cdot b := a_{(1)}b$ for $a, b \in V_2$ forms a commutative algebra with an invariant bilinear form $\langle \cdot, \cdot \rangle$ defined by $\langle a, b \rangle \mathbf{1} = a_{(3)}b$. This algebra is called the *Griess algebra* of V . Let e, f be Ising vectors of V of σ -type. Then $e/2$ and $f/2$ are idempotents of the Griess algebra V_2 with squared norm $1/16$. The following result is fundamental:

Proposition 3.4 ([26, 28]) *Let e, f be distinct Ising vectors of V of σ -type. Then one of the following holds:*

- (1) $\langle e, f \rangle = 0$ and $e \cdot f = 0$. In this case $\sigma_e f = f$ and $\sigma_f e = e$.
- (2) $\langle e, f \rangle = 1/32$ and $e \neq \sigma_f e = \sigma_e f \neq f$. In this case $\sigma_e \sigma_f$ is of order 3 and the following equality holds in V_2 :

$$e \cdot f = \frac{1}{4}(e + f - \sigma_e f). \quad (3.5)$$

Therefore, e and f generates a three dimensional subalgebra $\mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}\sigma_e f$ in the Griess algebra V_2 on which the symmetric group of degree three acts.

Let H_8 be the $[8, 4, 4]$ -Hamming code:

$$H_8 := \text{Span}_{\mathbb{Z}_2} \{ (11111111), (11110000), (11001100), (10101010) \} \subset \mathbb{Z}_2^8.$$

It is well known that H_8 is the unique doubly even self-dual code of length 8. Let V_{H_8} be the code VOA associated to H_8 and let $\omega = e^1 + \cdots + e^8$ be the standard Ising

¹ The signs depend on the choice of 2-cocycle which we have use to construct a tensor product $X^{\otimes n}$.

frame of V_{H_8} . Inside the Hamming code VOA V_{H_8} , we can find three Ising frames. For a codeword $\alpha \in \mathbb{Z}_2^8$, define

$$t^\alpha := \frac{1}{8} \sum_{i=1}^8 e^i + \frac{1}{8} \sum_{\beta \in H_8, \langle \beta, \beta \rangle = 4} (-1)^{\langle \alpha, \beta \rangle} x^\beta \in V_{H_8}, \quad (3.6)$$

where x^β above are defined as in (3.4). Then it is shown in [29] that t^α is an Ising vector of σ -type.

Proposition 3.5 ([26, 31]) *Inside the Hamming code VOA V_{H_8} , there are exactly three Ising frames given as follows.*

$$I_0 := \{e^i \mid 1 \leq i \leq 8\}, \quad I_1 := \{t^{v^j} \mid 1 \leq j \leq 8\} \text{ and } I_2 := \{t^{v^1+v^k} \mid 1 \leq k \leq 8\}, \quad (3.7)$$

where we have set $v^1 := (10000000)$, $v^2 := (01000000)$, $\dots$, $v^8 := (00000001) \in \mathbb{Z}_2^8$. Moreover, if $f \in I_a$, then $\sigma_f I_b = I_c$ if $\{a, b, c\} = \{0, 1, 2\}$ so that all the frames are mutually conjugate to each others.

Based on Propositions 3.4 and 3.5, the following result is established in [17].

Proposition 3.6 ([17]) *Let C be an even linear code whose minimum weight is greater than 2. If the code VOA V_C contains an Ising vector f of σ -type which is not a summand of the standard Ising frame of V_C , then C contains a subcode D isomorphic to H_8 such that $f \in V_D \subset V_C$ and f is of the form (3.6) in V_D . In particular, there are exactly 24 Ising vectors inside V_{H_8} and every Ising vector of V_{H_8} is a summand of an Ising frame of V_{H_8} .*

We give a generalization of the proposition above. The following lemma enables us to reduce a general case to the known case.

Lemma 3.7 *Let V be a VOA with $V_1 = 0$. Suppose that V has two Ising vectors e, f and e is of σ -type. Then $e \in V^{(\tau_f)}$.*

Proof Consider the Griess algebra V_2 of V . Let t be an Ising vector of V . Then the grade preserving operator $t_{(1)}$ acts on V_2 semisimply. Set $B_t(h) := \{v \in V_2 \mid t_{(1)}v = hv\}$ for $h = 0, 1/2, 1/16$. Then it is shown in [28] that the Griess algebra V_2 affords the orthogonal decomposition

$$V_2 = \mathbb{C}t \perp B_t(0) \perp B_t(1/2) \perp B_t(1/16). \quad (3.8)$$

For $x \in V_2$, we will denote the corresponding decomposition by $x = \lambda_{x,t}t + x_t(0) + x_t(1/2) + x_t(1/16)$ with $\lambda_{x,t} \in \mathbb{C}$. Since $\langle x, t \rangle = \lambda_{x,t}\langle t, t \rangle = \lambda_{x,t}/4$, the scalar $\lambda_{x,t}$ is given by $4\langle x, t \rangle$. By the fusion rules of $L(1/2, 0)$ -modules (cf. [9]), V_2 has the following structure:

$$\begin{aligned} B_t(0) \cdot B_t(h) &\subset B_t(h), \quad h = 0, 1/2, 1/16, & B_t(1/2) \cdot B_t(1/2) &\subset \mathbb{C}t \oplus B_e(0), \\ B_t(1/2) \cdot B_t(1/16) &\subset B_t(1/16), & B_t(1/16) \cdot B_t(1/16) &\subset \mathbb{C}t \oplus B_t(0) \oplus B_t(1/2). \end{aligned} \quad (3.9)$$

Write $f = \lambda e + f_e(0) + f_e(1/2)$ and $e = \lambda f + e_f(0) + e_f(1/2) + e_f(1/16)$ with $\lambda = 4\langle e, f \rangle$. We shall show that $e_f(1/16) = 0$. Since f is an Ising vector, we have $f \cdot f = 2f$. Using (3.9), we compare the $V_e(1/2)$ -parts of both sides of $f \cdot f = 2f$ and obtain

$$f_e(0) \cdot f_e(1/2) = \frac{1}{2}(2 - \lambda)f_e(1/2). \quad (3.10)$$

Similarly, we compare the $V_f(l/16)$ -parts in the equality $e \cdot e = 2e$ and get

$$e_f(0) \cdot e_f(l/16) + e_f(l/2) \cdot e_f(l/16) = \frac{1}{16}(16 - \lambda)e_f(l/16). \quad (3.11)$$

Since the Griess algebra V_2 is commutative, $e \cdot f = f \cdot e$ and we have

$$f_e(l/2) = 4\lambda(1 - \lambda)f - 4\lambda e_f(0) + (1 - 4\lambda)e_f(l/2) + \frac{1}{8}(1 - 32\lambda)e_f(l/16). \quad (3.12)$$

Using $f_e(0) = f - \lambda e - f_e(l/2)$, we have

$$f_e(0) = (3\lambda - 1)(\lambda - 1)f + 3\lambda e_f(0) + (3\lambda - 1)e_f(l/2) + \frac{1}{8}(24\lambda - 1)e_f(l/16). \quad (3.13)$$

From the $V_f(l/16)$ -part in the equality $e \cdot f_e(0) = 0$ we obtain

$$e_f(l/2) \cdot e_f(l/16) = \frac{1}{16}(92\lambda - 1)e_f(l/16). \quad (3.14)$$

Then by (3.11) we have

$$e_f(0) \cdot e_f(l/16) = \frac{1}{16}(-93\lambda + 17)e_f(l/16). \quad (3.15)$$

By (3.14), (3.15) and the $V_f(l/16)$ -parts in the equality (3.10), we obtain

$$\frac{1}{128}(32\lambda - 1)(64\lambda + 13)e_f(l/16) = 0. \quad (3.16)$$

Therefore, either $\lambda = 1/32$, $\lambda = -13/64$ or $e_f(l/16) = 0$.

Now suppose that $e_f(l/16) \neq 0$. Then $\lambda = 1/32$ or $-13/64$. By comparing the $V_f(l/2)$ -parts in the equality $e \cdot e = 2e$ we obtain

$$(e_f(l/16) \cdot e_f(l/16))_f(l/2) = (2 - \lambda)e_f(l/2) - 2e_f(0) \cdot e_f(l/2). \quad (3.17)$$

Then from the $V_f(l/2)$ -part in the equality $e \cdot f_e(0) = 0$ together with (3.17) we get

$$e_f(0) \cdot e_f(l/2) = \frac{1}{6}(29\lambda + 2)e_f(l/2) \quad (3.18)$$

and from the $V_f(l/2)$ -parts in the equality (3.10) we also have

$$\frac{1}{32}(168\lambda + 1)e_f(0) \cdot e_f(l/2) = \frac{1}{64}(600\lambda^2 - 17\lambda + 34)e_f(l/2). \quad (3.19)$$

Since $\lambda = 1/32$ or $-13/64$, the equalities (3.18) and (3.19) imply that $e_f(l/2) = 0$. Then Eq. (3.14) contradicts to our assumption. Thus $e_f(l/16) = 0$ and the lemma follows. $\square$

By the lemma above, we can generalize Proposition 3.6 as follows.

Proposition 3.8 *Let C be an even linear code with minimum weight greater than 2, and let f be an Ising vector of the associated code VOA V_C . If f is not a summand of the standard Ising frame of V_C , then there is a subcode D of C isomorphic to H_8 such that $f \in V_D \subset V_C$ and f is of the form (3.6) in V_D .*

Proof Let $\omega = e^1 + \cdots + e^n$ be the standard Ising frame of V_C . Then by Lemma 3.7 all e^i are contained in the τ_f -fixed point subalgebra $V_C^{(\tau_f)}$ of V_C . Therefore, there is a subcode C' of C with index at most two such that $V_C^{(\tau_f)}$ is a code VOA $V_{C'}$ with respect to the Ising frame $\omega = e^1 + \cdots + e^n$. Since f is of σ -type on $V_{C'}$, we can apply Proposition 3.6 to $V_{C'}$. This completes the proof. $\square$

Corollary 3.9 *Let C be an even linear code of length n and assume that the minimum weight of C is greater than 2.*

- (1) *Every Ising vector of V_C is a summand of an Ising frame of V_C .*
- (2) *Let N be the number of embeddings of H_8 into C . Then V_C contains exactly $16N + n$ Ising vectors.*

As an application of Proposition 3.6, we show the conjugacy property of Ising frames of a code VOA.

Lemma 3.10 *Let C be an even linear code of length n with no weight two element. Suppose that there is a subcode D of C which is isomorphic to the Hamming code H_8 . Then an Ising vector $t \in V_D \subset V_C$ of the form (3.6) is of σ -type if and only if $|\text{supp}(\alpha) \cap \text{supp}(D)|$ is even for all codeword $\alpha \in C$.*

Proof We may assume that $\text{supp}(D) = \{1, \dots, 8\}$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a codeword of C . Then V_C is a sum of a coset type V_D -submodules $V_{D+(\alpha_1, \dots, \alpha_8)}$ as a V_D -module. It is shown in Theorem 2.2 of [31] that τ_t is trivial on $V_{D+(\alpha_1, \dots, \alpha_8)}$ if and only if $(\alpha_1, \dots, \alpha_8)$ is an even codeword. So the assertion follows. $\square$

By Proposition 3.6 and the lemma above, we can in principle count the number of σ -type Ising frames of V_C . Denote by G the subgroup of $\text{Aut}(V_C)$ generated by σ -type Miyamoto involutions. By Proposition 3.4, G is a 3-transposition group (see Definition 6.1).

Proposition 3.11 *Let C be an even linear code of length n whose minimum weight is greater than 2 and let $\omega = e^1 + \cdots + e^n$ and $\omega = f^1 + \cdots + f^n$ be σ -type Ising frames of the code VOA V_C . Then there is an element $\rho \in G$ such that $\{\rho e^1, \dots, \rho e^n\} = \{f^1, \dots, f^n\}$.*

Proof It is enough to show the assertion in the case that $\{e^1, \dots, e^n\}$ is the standard Ising frame of V_C . Set $I = \{e^1, \dots, e^n\}$ and $J = \{f^1, \dots, f^n\}$. We shall prove the assertion inductively. Assume that there is an element $\rho_i \in G$ such that $\{f^1, \dots, f^i\} \subset \rho_i I$, where in the case of $i = 0$ we set $\rho_0 = \text{id}$. Since I and $\rho_i I$ are conjugate, the associated binary code of V_C with respect to the frame $\rho_i I$ is still isomorphic to C . Therefore, by replacing I by $\rho_i I$, we may assume that $\{f^1, \dots, f^i\} \subset I \cap J$. If $f^{i+1} \notin I$, then there is a subcode D of C isomorphic to the Hamming code H_8 such that $f^{i+1} \in V_D \subset V_C$ and f^{i+1} is of the form (3.6) in V_D by Proposition 3.6. Let $\text{supp}(D) = \{j_1, \dots, j_8\}$. Then $\{\sigma_{e^{j_1}} f^{i+1}, \dots, \sigma_{e^{j_8}} f^{i+1}\} = \{\sigma_{f^{j_1+1}} e^{j_1}, \dots, \sigma_{f^{j_8+1}} e^{j_8}\}$ by Proposition 3.5. Therefore, $f^{i+1} \in \sigma_{e^{j_1}} \sigma_{f^{j_1+1}} I$. It follows from (3.6) that $\langle f^{i+1}, e^j \rangle = 1/32$ for $j \in \text{supp}(D)$. Hence $\{f^1, \dots, f^i\} \cap \{e^{j_1}, \dots, e^{j_8}\} = \emptyset$ and $\sigma_{e^{j_1}} \sigma_{f^{j_1+1}} \{f^1, \dots, f^i\} = \{f^1, \dots, f^i\}$. Thus $\rho_{i+1} = \sigma_{e^{j_1}} \sigma_{f^{j_1+1}} \in G$ satisfies $\{f^1, \dots, f^{i+1}\} \subset \rho_{i+1} I$. By this procedure, we will obtain $\rho = \rho_n \in G$ such that $\rho I = J$. $\square$

Remark 3.12 We can slightly generalize Proposition 3.11 as follows. Let $\omega = e^1 + \cdots + e^n$ be a σ -type Ising frame of V_C and $\omega = f^1 + \cdots + f^n$ any Ising frame of V_C . Let H be the subgroup of $\text{Aut}(V_C)$ generated by τ -type Miyamoto involutions $\{\tau_{f^i} \mid 1 \leq i \leq n\}$. Then f^i are σ -type Ising vectors on V_C^H . By Lemma 3.7, $\{e^1, \dots, e^n\}$ is contained in V_C^H . Therefore, by Proposition 3.11, we can find $\rho \in \text{Aut}(V_C^H)$ such that $\rho\{e^1, \dots, e^n\} = \{f^1, \dots, f^n\}$. However, we cannot find such ρ inside $\text{Aut}(V_C)$ unless $H = 1$. In fact, conjugating VOA structures by ρ , we can perform a $\mathbb{Z}_2$ -twisted orbifold construction of a framed VOA (cf. [32, 36]).

We have only considered code VOAs associated to codes without weight two elements. If an even linear code C contains a weight two element, then the code VOA V_C contains a subalgebra isomorphic to a lattice VOA $V_{\mathbb{Z}\alpha}$ associated to a lattice $\mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 4$ (cf. [9]). In this case, we can define continuous automorphisms on V_C by exponential and hence $\text{Aut}(V_C)$ is always an infinite group. Conversely, if C contains no weight two element, then $\text{Aut}(V_C)$ is finite. This result is established in [29] in the case that V_C is considered over $\mathbb{R}$. By Proposition 3.6, we know that the set of σ -type Ising vectors of V_C is finite so that $\text{Aut}(V_C)$ is still finite even if V_C is defined over $\mathbb{C}$. We give a brief description of $\text{Aut}(V_C)$ as follows.

Proposition 3.13 *Let C be an even linear code of length n without weight two codewords. Then the automorphism group $\text{Aut}(V_C)$ of the code VOA V_C is a finite group generated by the lift of $\text{Aut}(C)$ and σ -type Miyamoto involutions.*

Proof Let $I = \{e^1, \dots, e^n\}$ be the standard Ising frame of V_C and let G be the 3-transposition subgroup of $\text{Aut}(V_C)$ generated by σ -type Miyamoto involutions. Take any $\phi \in \text{Aut}(V_C)$. By Proposition 3.11, there is an element $\rho \in G$ such that $\rho\phi I = I$. Then $\rho\phi$ defines an automorphism of $\text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n)$. Therefore, $\rho\phi$ preserves the set $\{\pm x^\alpha \in V_C \mid \alpha \in C\}$ of normed highest weight vectors and thus there is a lift $\tilde{g} \in \text{Aut}(V_C)$ of $g \in \text{Aut}(C)$ such that $\tilde{g}\rho\phi e^i = e^i$ for $i = 1, \dots, n$. Then by Schur's lemma, $\tilde{g}\rho\phi$ is written as a product of σ_{e^i} , $1 \leq i \leq n$. Thus $\text{Aut}(V_C)$ is generated by G and the lift of $\text{Aut}(C)$ on V_C . Since there are finitely many Ising frames inside V_C , the argument above also shows that $\text{Aut}(V_C)$ is finite. $\square$

Let E be the set of σ -type Ising vectors of V . We have defined a map $\sigma : E \rightarrow \text{Aut}(V)$ by associating the σ -type Miyamoto involution $\sigma_e \in \text{Aut}(V)$ to each $e \in E$. The following injectivity is shown in [25].²

Lemma 3.14 ([25]) *Assume that for each $e \in E$, there is an Ising vector $g \in V$ such that $\langle e, g \rangle = 1/32$. Then $\sigma : E \rightarrow \text{Aut}(V)$ is injective.*

Proof Suppose $\sigma_e = \sigma_f$ with $e, f \in E$. By the assumption, there exists an Ising vector $g \in V$ such that $\langle e, g \rangle = 1/32$. It follows from Lemma 3.7 that $e, f \in V^{\langle \tau_g \rangle}$. Then by (2) of Proposition 3.4, $g \neq \sigma_e g = \sigma_g e \neq e$ and hence $g \neq \sigma_g e = \sigma_e g = \sigma_f g$. Thus $\langle f, g \rangle = 1/32$ and $\sigma_f g = \sigma_g f$ again by Proposition 3.4. Hence $\sigma_g e = \sigma_e g = \sigma_f g = \sigma_g f$ showing $e = f$. $\square$

² Lemma 2.5.2 in the preprint version.

4 Ising vectors of lattice VOAs

We will classify Ising vectors of the $\mathbb{Z}_2$ -orbifold $V_{\sqrt{2}R}^+$ of a lattice $V_{\sqrt{2}R}$ associated with a root lattice R . This classification immediately leads to a classification of Ising vectors in M_R .

4.1 A lattice VOA and its code

Let $\mathbb{Z}^n$ be the standard lattice and let $\rho : \mathbb{Z}^n \rightarrow \mathbb{Z}_2^n$ be the reduction mod 2, which is a group homomorphism. For an even linear code $C \subset \mathbb{Z}_2^n$, the preimage $L_C := \rho^{-1}(C) \subset \mathbb{Z}^n$ define a sublattice of $\mathbb{Z}^n$. Set $x^1 := (1, 0, \dots, 0)$, $x^2 := (0, 1, \dots, 0)$, $\dots$, $x^n := (0, 0, \dots, 1) \in \mathbb{Z}^n$. Then $\{x^1, \dots, x^n\}$ is the standard basis of $\mathbb{Z}^n$. By definition L_C contains a 4-frame $2\mathbb{Z}x^1 \perp \dots \perp 2\mathbb{Z}x^n$ so that V_{L_C} has a full sub VOA isomorphic to $V_{2\mathbb{Z}x^1} \otimes \dots \otimes V_{2\mathbb{Z}x^n}$. In the lattice VOA $V_{2\mathbb{Z}x^i}$, set

$$w^{i\pm} := \frac{1}{4} \left(x_{(-1)}^i \right)^2 \mathbf{1} \pm \frac{1}{4} \left(e^{2x^i} + e^{-2x^i} \right). \quad (4.1)$$

Then $w^{i\pm}$ are mutually orthogonal Ising vectors. Let ω be the Virasoro vector of V_{L_C} . The lattice VOA V_{L_C} contains an Ising frame

$$W = \left(w^{1-} + w^{1+} \right) + \left(w^{2-} + w^{2+} \right) + \dots + \left(w^{n-} + w^{n+} \right). \quad (4.2)$$

It is easy to see that all $w^{i\pm}$ are of σ -type on V_{L_C} so that V_{L_C} is a code VOA with respect to the frame. It is obvious that $w^{i\pm}$ are contained in $V_{2\mathbb{Z}x^i}^+$ so that they are also contained in $V_{L_C}^+$. We define

$$\begin{aligned} D_0(C) &:= \{(u_1, u_1, u_2, u_2, \dots, u_n, u_n) \in \mathbb{Z}_2^{2n} \mid (u_1, u_2, \dots, u_n) \in C^\perp\} \subset \mathbb{Z}_2^{2n}, \\ D_1(C) &:= D_0(C) \cup (D_0(C) + \gamma) \subset \mathbb{Z}_2^{2n}, \quad \gamma = (1010 \dots 10) \in \mathbb{Z}_2^{2n}. \end{aligned}$$

As $\text{Vir}(w^{i-}) \otimes \text{Vir}(w^{i+})$ -modules, we have the following isomorphisms:

$$\begin{aligned} V_{2\mathbb{Z}x^i}^+ &\simeq L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0), & V_{2\mathbb{Z}x^i}^- &\simeq L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}), \\ V_{2\mathbb{Z}x^i+x^i}^+ &\simeq L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2}), & V_{2\mathbb{Z}x^i+x^i}^- &\simeq L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0). \end{aligned}$$

By the isomorphisms above, the structure codes of V_{L_C} and $V_{L_C}^+$ with respect to the frame (4.2) are described as follows.

Proposition 4.1

- (1) The lattice VOA V_{L_C} with the Ising frame (4.2) is isomorphic to the code VOA associated to $D_0(C)^\perp$.
- (2) The VOA $V_{L_C}^+$ with the Ising frame (4.2) is isomorphic to the code VOA associated to $D_1(C)^\perp$.

If we take

$$C_n := \text{Span}_{\mathbb{Z}_2} \{ (1^2 0^{2n-2}), (0^2 1^2 0^{2n-4}), \dots, (0^{2n-2} 1^2), (1010 \dots 10) \} \subset \mathbb{Z}_2^{2n},$$

then we obtain $L_{C_n^\perp} \simeq \sqrt{2}D_{2n}$. Therefore, the structure code of $V_{\sqrt{2}D_{2n}}$ as a code VOA is given by the dual of the following code:

$$C_n := \text{Span}_{\mathbb{Z}_2} \{ (1^4 0^{4n-4}), (0^4 1^4 0^{4n-8}), \dots, (0^{4n-4} 1^4), (1^2 0^2 \dots 1^2 0^2), \} \subset \mathbb{Z}_2^{4n}, \quad (4.3)$$

and that of $V_{\sqrt{2}D_{2n}}^+$ is given by the dual code of the code generated by C_n and $(1010 \dots 10) \in \mathbb{Z}_2^{4n}$.

If we take the $[8,4,4]$ -Hamming code H_8 , then $H_8^\perp = H_8$ and we obtain $L_{H_8} \simeq \sqrt{2}E_8$. The structure code of $V_{\sqrt{2}E_8}^+$ as a code VOA is given by $D_1(H_8)^\perp$, where $D_1(H_8)$ has the following presentation:

$$D_1(H_8) = \text{Span}_{\mathbb{Z}_2} \left\{ (1^{16}), (1^8 0^8), (1^4 0^4 1^4 0^4), (1^2 0^2 1^2 0^2 1^2 0^2), (\{10\}^8) \right\} \subset \mathbb{Z}_2^{16}.$$

We note that the code $D_1(H_8)$ is the same as the first order Reed–Muller code $\text{RM}(1, 4)$ of length 2^4 so that the code $D_1(H_8)^\perp$ is equal to the second order Reed–Muller code $\text{RM}(2, 4)$ (cf. [4]).

4.2 Ising vectors of $V_{\sqrt{2}R}^+$

Let R be an indecomposable root lattice with root system $\Phi(R)$. We give a classification of Ising vectors of $V_{\sqrt{2}R}^+$. As in (4.1), for $\alpha \in \Phi(R)$ we set

$$w^\pm(\alpha) := \frac{1}{8} \alpha_{(-1)}^2 \mathbf{1} \pm \frac{1}{4} \left(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha} \right) \in V_{\sqrt{2}R}^+. \quad (4.4)$$

It is shown in [7,9] that $w^\pm(\alpha)$, $\alpha \in \Phi(R)$, are Ising vectors of σ -type.

Proposition 4.2 *Let R be an indecomposable root lattice whose root system is of type A_n or D_n . Then the set $\{w^\pm(\alpha) \mid \alpha \in \Phi(R)\}$ exhausts all the Ising vectors of $V_{\sqrt{2}R}^+$.*

Proof It suffices to show the assertion in the case of $R = D_{2n}$ since A_k and D_{2m+1} , $1 \leq k, m \leq n-1$, are sublattices of D_{2n} . Let $\mathcal{C}$ be the dual code of the following code:

$$\text{Span}_{\mathbb{Z}_2} \{ (1^4 0^{4n-4}), (0^4 1^4 0^{4n-8}), \dots, (0^{4n-4} 1^4), (1^2 0^2 \dots 1^2 0^2), (1010 \dots 1010) \} \subset \mathbb{Z}_2^{4n}.$$

As shown in Sect. 4.1, $V_{\sqrt{2}D_{2n}}^+$ is isomorphic to a code VOA associated to $\mathcal{C}$. By Corollary 3.9, it suffices to count the number N of embeddings of H_8 into $\mathcal{C}$ and one can easily show that $N = \binom{n}{2} = n(n-1)/2$. Therefore, there are $4n(2n-1)$ Ising vectors inside $V_{\sqrt{2}D_{2n}}^+$. Since there are $|\Phi(D_{2n})| = 4n(2n-1)$ Ising vectors of the form (4.4) in $V_{\sqrt{2}D_{2n}}^+$, they are all. $\square$

Next we consider the E -series. Since E_6 and E_7 are sublattices of E_8 , we only need to consider $V_{\sqrt{2}E_8}^+$. Since the number of roots in $\Phi(E_8)$ is 240, we have 240 Ising vectors of $V_{\sqrt{2}E_8}^+$ of the form (4.4). We also have an Ising vector $\tilde{\omega}$ of $V_{\sqrt{2}E_8}^+$ defined as in (2.3). Since $\tau_{\tilde{\omega}} = \text{id}$ on $V_{\sqrt{2}E_8}^+$, $\tilde{\omega}$ is of σ -type. For $h \in \mathbb{C}E_8$, set

$$\varphi_h := \exp \left(\frac{\pi \sqrt{-2}}{2} (h_{(-1)} \mathbf{1})_{(0)} \right) \in \text{Aut} \left(V_{\sqrt{2}E_8} \right). \quad (4.5)$$

Then $\varphi_h V_{\sqrt{2}E_8}^+ = V_{\sqrt{2}E_8}^+$ if and only if $\varphi_{2h} = 1$. Therefore, we obtain a group homomorphism $\varphi : E_8 \ni x \mapsto \varphi_x \in \text{Aut}(V_{\sqrt{2}E_8}^+)$ with $\ker \varphi = 2E_8$. So we have $2^8 = 256$ Ising vectors of $V_{\sqrt{2}E_8}^+$ of the form

$$\varphi_x \tilde{\omega}, \quad x \in E_8. \quad (4.6)$$

It is shown in [4] that $E_8/2E_8$ contains 1 class represented by 0, 120 classes represented by a pair of roots $\pm\alpha \in \Phi(E_8)$, and 135 classes represented by 16 vectors forming a 4-frame of E_8 .

It is shown by Shimakura [34] that these $240 + 256 = 496$ vectors exhaust all the Ising vectors of $V_{\sqrt{2}E_8}^+$. We give another proof here.

Proposition 4.3 *There are 496 Ising vectors inside $V_{\sqrt{2}E_8}^+$.*

Proof As shown in Sect. 4.1, there is an Ising frame of $V_{\sqrt{2}E_8}^+$ such that $V_{\sqrt{2}E_8}^+$ is isomorphic to a code VOA associated to the Reed–Muller code $\text{RM}(2, 4)$ with respect to the frame.

The minimum weight of $\text{RM}(2, 4)$ is 4 (cf. [4]). So by Corollary 3.9, it suffices to count embeddings of the $[8, 4, 4]$ -Hamming code H_8 to $\text{RM}(2, 4)$. It is not difficult to see that for each embedding $H_8 \hookrightarrow \text{RM}(2, 4)$, the support of H_8 belongs to $\text{RM}(2, 4)^\perp = \text{RM}(1, 4)$ so that there are exactly 30 embeddings of H_8 into $\text{RM}(2, 4)$. Hence, there exists $30 \times 16 = 480$ Ising vectors of the form (3.6) inside $V_{\sqrt{2}E_8}^+$. Adding the 16 Ising vectors of summands of the standard Ising frame, there are $480 + 16 = 496$ Ising vectors which exhaust all the Ising vectors inside $V_{\sqrt{2}E_8}^+$. We also note that all the Ising vectors are of σ -type on $V_{\sqrt{2}E_8}^+$. $\square$

Combining Proposition 4.2 and 4.3, we obtain the following.

Theorem 4.4 *Let R be an indecomposable root lattice with root system $\Phi(R)$.*

- (1) *If $R \neq E_8$, the set $\{w^\pm(\alpha) \mid \alpha \in \Phi(R)\}$ exhausts all the Ising vectors of $V_{\sqrt{2}R}^+$.*
- (2) *There are 256 Ising vectors of $V_{\sqrt{2}E_8}^+$ other than $\{w^\pm(\alpha) \mid \alpha \in \Phi(E_8)\}$. The set $\{w^\pm(\alpha), \varphi_x \tilde{\omega} \mid \alpha \in \Phi(E_8), x \in E_8/2E_8\}$ exhausts all the Ising vectors of $V_{\sqrt{2}E_8}^+$.*

Corollary 4.5 *There is no τ -type Ising vector inside the $\mathbb{Z}_2$ -orbifold subalgebra $V_{\sqrt{2}R}^+$ of the lattice VOA $V_{\sqrt{2}R}$ associated to an indecomposable root lattice R .*

We have treated only indecomposable root lattices. Now we classify all Ising vectors of the $\mathbb{Z}_2$ -orbifold $V_{\sqrt{2}L}^+$ of a lattice VOA $V_{\sqrt{2}L}$ associated to any root lattice L .

Theorem 4.6 *Let L be a root lattice. If e is an Ising vector of $V_{\sqrt{2}L}^+$, then there exists a sublattice K of L isometric to A_1 or E_8 such that $e \in V_{\sqrt{2}K}^+ \subset V_{\sqrt{2}L}^+$.*

Proof Since L is a root lattice, L is a direct sum of indecomposable root lattices of ADE -type. Let $L^{(1)}$ be the sum of irreducible sublattices of L of AD -type and $L^{(2)}$ the orthogonal complement of $L^{(1)}$ in L which is the sum of irreducible sublattices of L of E -type. Let $L^{(2)} = L^{(2,1)} \oplus \cdots \oplus L^{(2,n)}$ be the decomposition of $L^{(2)}$ into irreducible components. Then there exists $N \in \mathbb{N}$ such that we can embed $L^{(1)}$ into D_{2N} and $L^{(2,i)}$ into E_8 for $1 \leq i \leq n$. Set $R^0 = D_{2N}$ and $R^i = E_8$ for $1 \leq i \leq n$. By using the embeddings above, we identify L as a sublattice of $R^0 \oplus R^1 \oplus \cdots \oplus R^n \simeq D_{2N} \oplus E_8^{\oplus n}$ by which we regard $V_{\sqrt{2}L}^+$ as a sub-VOA of $V_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}^+$. We shall show that if e is an Ising vector of $V_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}^+$, then e is contained in $V_{\sqrt{2}R^0}^+ \otimes \cdots \otimes V_{\sqrt{2}R^n}^+$. Recall the code C_N of length $4N$ defined by (4.3). Let C be the dual code of the code generated by

$\mathcal{C}_N \oplus D_0(H_8)^{\oplus n}$ and $\gamma = (1010 \dots 10) \in \mathbb{Z}_2^{4N+16n}$. As shown in Sec. 4.1, $V_{\sqrt{2}(R^0 \oplus \dots \oplus R^n)}^+$ is isomorphic to a code VOA V_C associated to the code C . Then by Proposition 3.8 there exists an embedding $\phi: H_8 \hookrightarrow C$ such that $e \in V_{\phi(H_8)} \subset V_C$. It is not difficult to verify that $\text{supp}(\phi(H_8))$ is contained in one of the direct summands of $\mathcal{C}_n^\perp \oplus (D_0(H_8)^\perp)^{\oplus n}$. This implies that e must be in $V_{\sqrt{2}R^0}^+ \otimes \dots \otimes V_{\sqrt{2}R^n}^+$ as we claimed. Then $e \in V_{\sqrt{2}R^i}^+$ for some $0 \leq i \leq n$. Therefore, by Theorem 4.4, either there is a root $\alpha \in R^0 \oplus \dots \oplus R^n$ such that e is of the form $w^\pm(\alpha)$ as in (4.4), or $e \in V_{\sqrt{2}R^i}^+$ for some $i > 0$ and e is of the form (4.6) in $V_{\sqrt{2}R^i}^+$. Anyway, there is a sublattice K of $R^0 \oplus \dots \oplus R^n$ isometric to A_1 or E_8 such that $e \in V_{\sqrt{2}K}^+ \subset V_{\sqrt{2}(R^0 \oplus \dots \oplus R^n)}^+$. If e is taken from $V_{\sqrt{2}L}^+$, then $K \subset L$ as we know the explicit form of e in $V_{\sqrt{2}(R^0 \oplus \dots \oplus R^n)}^+$. This completes the proof. $\square$

Remark 4.7 With reference to the theorem above, we believe that the same is true for the $\mathbb{Z}_2$ -orbifold V_L^+ of a lattice VOA associated with any even lattice L without roots. For example, it is shown by Shimakura [34] that this is true for the $\mathbb{Z}_2$ -orbifold V_Λ^+ of the Leech Lattice VOA V_Λ . However, we do not know the answer at present.

4.3 Ising vectors of M_R

Let R be an indecomposable root lattice as before. We shall determine the Ising vectors of M_R . Recall that

$$w^\pm(\alpha) = \frac{1}{8} \alpha_{(-1)}^2 \mathbf{1} \pm \frac{1}{4} \left(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha} \right), \quad \text{for } \alpha \in R,$$

and

$$\tilde{\omega} = \tilde{\omega}_R = \frac{2}{h+2} \omega + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha},$$

where h denotes the Coxeter number of $\Phi(R)$ as before. From the definition, it is easy to show the following.

Lemma 4.8 *Let R be a root lattice and consider $V_{\sqrt{2}R}^+$. Let $\alpha, \beta \in \Phi(R)$. Then*

- (1) $\langle w^+(\alpha), w^-(\beta) \rangle = 1/32$ if $\langle \alpha, \beta \rangle = \pm 1$ and 0 otherwise.
- (2) $\langle w^\pm(\alpha), w^\pm(\beta) \rangle = 1/4$ if $\alpha = \pm \beta$, $1/32$ if $\langle \alpha, \beta \rangle = \pm 1$ and 0 if $\langle \alpha, \beta \rangle = 0$.
- (3) $\langle w^-(\alpha), \tilde{\omega} \rangle = 0$ and $\langle w^+(\alpha), \tilde{\omega} \rangle = 1/(h+2)$, where h is the Coxeter number of $\Phi(R)$.
- (4) For $R = E_8$, let $x \in E_8$. Then we have $\langle \varphi_x \tilde{\omega}, \tilde{\omega} \rangle = 1/4$ if $x \in 2E_8$, $1/32$ if x is represented by a root in $E_8/2E_8$ and 0 otherwise.

Lemma 4.9 *Let R be an indecomposable root lattice. The weight two subspace of M_R is spanned by $\{w^-(\alpha) \mid \alpha \in \Phi(R)\}$.*

Proof We note that $M_R = \ker_{V_{\sqrt{2}R}^+}(\tilde{\omega}_{(1)})$ since $V_{\sqrt{2}R}^+$ is a completely reducible $\text{Vir}(\tilde{\omega})$ -module. It is clear that $\{w^-(\alpha) \mid \alpha \in \Phi(R)\}$ is a set of linearly independent vectors of the weight two subspace of M_R , thanks to (3) of Lemma 4.8. If $R = A_n$, it is shown in [7] that the set $\{w^\pm(\alpha) \mid \alpha \in \Phi(R)\}$ is a basis of the weight two subspace of $V_{\sqrt{2}R}^+$. Therefore, by (3) of Lemma 4.8, the assertion holds in this case. It is shown in [5] that

$M_{D_n} \simeq V_{\sqrt{2}A_{n-1}}^+$. By Theorem 4.4 and (3) of Lemma 4.8, the assertion also holds if $R = D_n$.

Consider the case that $R = E_6$ or E_7 . The vacuum character of M_R is computed in Sect. 7 and one can verify that $\dim(M_R)_2 = |\Phi(R)|/2$. So the assertion follows. If $R = E_8$, then M_{E_8} is a code VOA and it is easy to know the structure code from which we can compute the vacuum character. As a result, one also has $\dim(M_{E_8})_2 = |\Phi(E_8)|/2$. This completes the proof. $\square$

5 Automorphism group of M_R

We will determine the automorphism group of the commutant subalgebra M_R where R is an indecomposable root lattice. It is shown that $\text{Aut}(M_R)$ always contains a half of the Weyl group $W(R)$ of R .

5.1 $\text{Aut}(M_R)$: the case $R \neq E_8$

Let R be an indecomposable root lattice. We suppose that R is not the E_8 -lattice. In this case, due to Theorem 4.4 and Lemma 4.8, the set of Ising vectors of M_R is given by $E_R := \{w^-(\alpha) \mid \alpha \in \Phi(R)\}$. We have shown in Lemma 4.9 that the weight two subspace of M_R is spanned by E_R . In fact, M_R is generated by E_R as a VOA.

Proposition 5.1 *M_R is generated by its weight two subspace.*

Proof The assertion is already shown in [18] if $R = A_n$. Since $M_{D_n} \simeq V_{\sqrt{2}A_{n-1}}^+$ by [5], the assertion is also true if $R = D_n$. So we only need to show the cases for $R = E_6$ and $R = E_7$. The proof is rather technical so that it will be given in Sect. 7. $\square$

Denote by $\Omega(E_R)$ the permutation group on E_R and we define

$$\text{Aut}(E_R) := \{\rho \in \Omega(E_R) \mid \langle \rho e, \rho f \rangle = \langle e, f \rangle \text{ for all } e, f \in E_R\}.$$

Since M_R is generated by E_R as a VOA, by restriction map we have an injection from $\text{Aut}(M_R)$ to $\text{Aut}(E_R)$. On the other hand, $g \in \text{Aut}(R)$ acts on E_R by $gw^-(\alpha) := w^-(g\alpha)$. Hence we also have a group homomorphism $\phi : \text{Aut}(R) \rightarrow \text{Aut}(E_R)$.

Lemma 5.2 *ϕ is surjective.*

Proof Let $\rho \in \text{Aut}(E_R)$. Take a simple system $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ of $\Phi(R)$, where ℓ is the rank of R . By Proposition 3.4 and Lemma 4.8, one can easily verify that $\{w^-(\alpha_1), \dots, w^-(\alpha_\ell)\}$ is a set of generators of the Griess algebra of M_R . Write $\rho w^-(\alpha_i) = w^-(\beta_i)$ for $i = 1, \dots, \ell$. By Proposition 3.4 and Lemma 4.8, there is a simple system Δ' of $\Phi(R)$ such that $\{\pm\beta_1, \dots, \pm\beta_\ell\} = \Delta' \cup (-\Delta')$. Therefore, by choosing suitable signs, we can take the representatives $\{\beta_i \mid 1 \leq i \leq \ell\}$ to be a simple system of $\Phi(R)$. Then we can find $g \in \text{Aut}(R)$ such that $g\alpha_i = \beta_i$ for $i = 1, \dots, \ell$. This implies that $\rho = g$ since $\{w^-(\alpha_i) \mid 1 \leq i \leq \ell\}$ is a set of generators of $(M_R)_2 = \text{Span}_{\mathbb{C}} E_R$. This completes the proof. $\square$

By definition, it is clear that $\ker \phi = \langle \pm 1 \rangle$. Therefore, we have

Proposition 5.3 $\text{Aut}(E_R) \simeq \text{Aut}(R)/\langle \pm 1 \rangle$.

The Weyl group $W(R)$ also acts on E_R . Let us denote by r_α the reflection on R defined by a root $\alpha \in \Phi(R)$. It follows from Proposition 3.4 and Lemma 4.8 that $\sigma_{w^-(\alpha)} w^-(\beta) = w^-(r_\alpha \beta)$ for $\alpha, \beta \in \Phi(R)$. Therefore, the group generated by $\{\sigma_e \mid e \in E_R\}$ realizes the action of the Weyl group on E_R . The shapes of the Weyl group $W(R)$ and the automorphism group $\text{Aut}(R)$ are as follows (cf. [4]).

R	$W(R)$	$\text{Aut}(R)$
A_n	S_{n+1}	$S_{n+1} \times 2$
D_4	$2^3 : S_4$	$(2^3 : S_4) : S_3$
D_n ($n > 4$)	$2^{n-1} : S_n$	$2^{n-1} : S_n : 2$
E_6	$U_4(2) : 2$	$2.U_4(2) : 2$
E_7	$2 \times \text{Sp}_6(2)$	$2 \times \text{Sp}_6(2)$

Theorem 5.4 *Suppose $R \neq E_8$. Then $\text{Aut}(M_R)$ is as follows*

R	A_1	A_n ($n > 1$)	D_4	D_n ($n > 4$)	E_6	E_7
$\text{Aut}(M_R)$	1	S_{n+1}	$(2^2 : S_4) : S_3$	$2^{n-1} : S_n$	$U_4(2) : 2 \simeq O_6^-(2)$	$\text{Sp}_6(2)$

Proof Let R be either A_n , $n > 1$, E_6 or E_7 . Under the injection, we can consider $\text{Aut}(M_R)$ as a subgroup of $\text{Aut}(E_R)$. By Proposition 5.3, we have the following relation.

$$\text{Aut}(M_R) \hookrightarrow \text{Aut}(E_R) \simeq \text{Aut}(R)/\langle \pm 1 \rangle.$$

Since the action of the Weyl group on E_R is realized by σ -type involutions $\{\sigma_e \mid e \in E_R\}$, the subgroup of $\text{Aut}(M_R)$ generated by $\{\sigma_e \mid e \in E_R\}$ coincides with $\text{Aut}(R)/\langle \pm 1 \rangle$. Hence $\text{Aut}(M_R) \simeq \text{Aut}(R)/\langle \pm 1 \rangle$. This establishes the cases (i), (iii) and (iv). The isomorphism in the case (ii) follows from the isomorphism $M_{D_n} \simeq V_{\sqrt{2}A_{n-1}}^+$ shown in [5] and the description of $\text{Aut}(V_{\sqrt{2}A_n}^+)$ obtained in [33]. The case $R = A_1$ is trivial since $M_{A_1} \simeq L(1/2, 0)$. This completes the proof. $\square$

5.2 $\text{Aut}(M_{E_8}) \simeq \text{Sp}_8(2)$

By Theorem 4.4, the set of Ising vectors of $V_{\sqrt{2}E_8}^+$ is given by

$$\{w^\pm(\alpha) \mid \alpha \in \Phi(E_8)\} \cup \{\varphi_x \tilde{\omega} \mid x \in E_8\}. \quad (5.1)$$

Lemma 5.5 *Let R be an indecomposable root lattice. Then all the Ising vectors of $V_{\sqrt{2}R}^+$ are conjugate under $\text{Aut}(V_{\sqrt{2}R}^+)$.*

Proof Assume $R \neq E_8$. Then the set of Ising vectors of $V_{\sqrt{2}R}^+$ is provided by $\{w^\pm(\alpha) \mid \alpha \in \Phi(R)\}$. It is clear that $V_{\sqrt{2}R}^+$ affords an action of the Weyl group $W(R)$ associated to the root system $\Phi(R)$ and $W(R)$ transitively acts on both $\{w^+(\alpha) \mid \alpha \in \Phi(R)\}$ and $\{w^-(\beta) \mid \beta \in \Phi(R)\}$ as R is indecomposable. If $R \neq A_1$, then by (2) of Proposition 3.4 and (1) of Lemma 4.8, there is a pair $\{w^+(\alpha), w^-(\beta)\}$ of Ising vectors which are conjugate under $\text{Aut}(V_{\sqrt{2}R}^+)$. Therefore, all the Ising vectors are conjugate. If $R = A_1$, then $R = \mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = 2$ and $V_{\sqrt{2}A_1}^+ \simeq \text{Vir}(w^-(\alpha)) \otimes \text{Vir}(w^+(\alpha))$. Since $g = \exp(\pi\sqrt{2}\alpha_{(0)}/4) \in \text{Aut}(V_{\sqrt{2}A_1}^+)$ permutes $w^-(\alpha)$ and $w^+(\alpha)$, they are conjugate, too.

Now assume that $R = E_8$. We have seen that $\text{Aut}(V_{\sqrt{2}E_8}^+)$ acts on $\{w^\pm(\alpha) \mid \alpha \in \Phi(E_8)\}$ transitively. It is obvious that Ising vectors of the form (4.6) are mutually

conjugate under $\text{Aut}(V_{\sqrt{2}E_8}^+)$. Then again by (2) of Proposition 3.4 and (3) of Lemma 4.8, all the Ising vectors of $V_{\sqrt{2}E_8}^+$ are conjugate. $\square$

As an immediate consequence, we have:

Corollary 5.6 *For any Ising vector e of $V_{\sqrt{2}E_8}^+$, $\text{Com}_{V_{\sqrt{2}E_8}^+}(\text{Vir}(e)) \simeq M_{E_8}$.*

Lemma 5.7 *M_{E_8} is generated by its weight two subspace.*

Proof It is shown in Sect. 4.1 that $V_{\sqrt{2}E_8}^+$ is isomorphic to the code VOA $V_{\text{RM}(2,4)}$ associated to the Reed–Muller code $\text{RM}(2, 4)$. By Corollary 5.6, we may assume that $\tilde{\omega}$ is the 1st summand of the standard Ising frame of $V_{\text{RM}(2,4)}$. Then M_{E_8} is also a code VOA and the associated code is obtained by collecting all the vectors of $\text{RM}(2, 4)$ whose 1st entry is 0 and dropping their 1st entry 0. It is easy to see that this code is generated by weight four vectors. Therefore, M_{E_8} is generated by its weight two subspace. $\square$

Lemma 5.8

- (1) *An Ising vector of M_{E_8} is equal to either $w^-(\alpha)$, $\alpha \in \Phi(E_8)$ or $\varphi_x \tilde{\omega}$, $x \in E_8$ and $x + 2E_8 \in E_8/2E_8$ is represented by a norm four vector of E_8 . Hence, there are $120 + 135 = 255$ Ising vectors of M_{E_8} .*
- (2) *These 255 Ising vectors are mutually conjugate under $\text{Aut}(M_{E_8})$.*

Proof

- (1) Since $V_{\sqrt{2}E_8}^+$ has a trivial weight one subspace, so does M_{E_8} . By (1) of Proposition 3.4, an Ising vector e of $V_{\sqrt{2}E_8}^+$ belongs to the commutant subalgebra $M_{E_8} = \text{Com}_{V_{\sqrt{2}E_8}^+}(\text{Vir}(\tilde{\omega}))$ if and only if $\langle e, \tilde{\omega} \rangle = 0$. So the assertion follows from Lemma 4.8.
- (2) It is clear that $w^-(\alpha)$, $\alpha \in \Phi(E_8)$, are mutually conjugate under $\text{Aut}(M_{E_8})$ since M_{E_8} affords a natural action of the Weyl group $W(E_8)$. One can also verify that for each $\varphi_x \tilde{\omega} \in M_{E_8}$, there is an Ising vector $w^-(\beta) \in M_{E_8}$ such that $\langle \varphi_x \tilde{\omega}, w^-(\beta) \rangle = 1/32$. Therefore, all the Ising vectors are mutually conjugate by (2) of Proposition 3.4. $\square$

Theorem 5.9 $\text{Aut}(M_{E_8}) \simeq \text{Sp}_8(2)$.

Proof Denote by E the set of Ising vectors of M_{E_8} and set

$$\sigma(E) := \{\sigma_e \in \text{Aut}(M_{E_8}) \mid e \in E\}.$$

Let G be the 3-transposition subgroup of $\text{Aut}(M_{E_8})$ generated by $\sigma(E)$. It is shown in [16, 25] that G is isomorphic to a simple group $\text{Sp}_8(2)$. Since E is invariant under $\text{Aut}(M_{E_8})$, $\sigma(E)$ is a normal set of $\text{Aut}(M_{E_8})$ and hence G is a normal subgroup of $\text{Aut}(M_{E_8})$. It is shown in [2] that $\text{Aut}(G) = \text{Inn}(G) \simeq G$. So the kernel of the conjugate action of $\text{Aut}(M_{E_8})$ on G is $C_{\text{Aut}(M_{E_8})}(G)$ and $\text{Aut}(M_{E_8}) \simeq G \times C_{\text{Aut}(M_{E_8})}(G)$. Let $g \in C_{\text{Aut}(M_{E_8})}(G)$. Since g commutes with $\sigma(E)$, g fixes E point-wise by Lemma 3.14. By Lemma 4.9, g is also the identity on the weight two subspace of M_{E_8} . Since M_{E_8} is generated by E as a VOA by Lemma 5.7, g is trivial on M_{E_8} . Thus $C_{\text{Aut}(M_{E_8})}(G) = 1$ and $\text{Aut}(M_{E_8}) = G \simeq \text{Sp}_8(2)$. $\square$

6 3-Transposition group and inductive structure

We will study a relation between an inductive structure of a 3-transposition group acting on a VOA and its commutant subalgebra. As an example, we study a commutant subalgebra of $V_{\sqrt{2}E_8}^+$ having $O_8^-(2)$ as its full automorphism group, which corresponds to an inductive structure $O_{10}^+(2)^{(2)} \simeq O_8^-(2)$. Note that $\text{Aut}(V_{\sqrt{2}E_8}^+) \simeq O_{10}^+(2)$ by [13, 33].

6.1 3-Transposition group

Definition 6.1 A (finite) 3-transposition group is a pair $(G, \mathcal{D})$ of a finite group G and a normal set $\mathcal{D}$ of involutions in G such that G is generated by $\mathcal{D}$ and if $x, y \in \mathcal{D}$ then the order of xy is either 1, 2 or 3.

A partial linear space consists of a set X called the set of points and a set $\mathcal{L}$ of subsets of X called the set of lines such that any two points lie on at most one line and any line has at least two points.

To a 3-transposition group $(G, \mathcal{D})$, we can associate a partial linear space called the Fischer space $(X, \mathcal{L})$ of $(G, \mathcal{D})$ as follows (cf. [25] and references therein). The set of points X is $\mathcal{D}$ and the set of lines $\mathcal{L}$ is such that a subset $\ell \subset \mathcal{D}$ is a line if and only if ℓ consists of three points which generate a subgroup of G isomorphic to S_3 . For $x, y \in \mathcal{D}$, we write $x \sim y$ if they lie on a line and $x \perp y$ if not.

The collinearity graph of $(X, \mathcal{L})$ is a graph Γ whose vertex set is X and $x, y \in X$ are adjacent if $x \sim y$, i.e., they are incident to a common line.

A 3-transposition group $(G, \mathcal{D})$ is referred to as of symplectic type if the affine plane of order 3 does not occur in the associated Fischer space. It is called indecomposable if the associated Fischer space has a connected collinearity graph.

Remark 6.2 Let ℓ_1 and ℓ_2 be two distinct lines with a common point in a Fischer space. It is known that the subspace generated by ℓ_1 and ℓ_2 is either isomorphic to a dual affine plane of order 2 or an affine plane of order 3 (cf. [1]). Thus if $(G, \mathcal{D})$ is of symplectic type, only the dual affine plane of order 2 can occur in the associated Fischer space. In this case, the subgroup generated by the corresponding involutions in ℓ_1 and ℓ_2 will be isomorphic to the symmetric group S_4 .

Let $(G, \mathcal{D})$ be a 3-transposition group and $(X, \mathcal{L})$ the associated Fischer space. Define $\sigma : X \rightarrow \text{Aut}(X)$ by $\sigma_x(y) := y^x$ for $x \in X$, and for $x_1, x_2, \dots \in X$, denote by

$$X_{x_1, x_2, \dots} := \{x \in X \mid x \perp x_i, i = 1, 2, \dots\}.$$

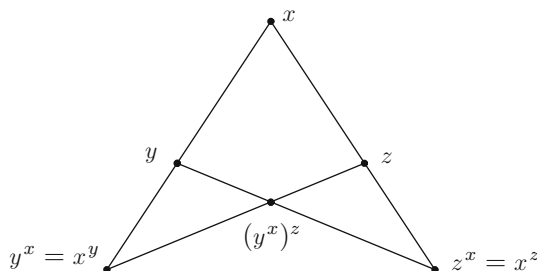
Since G is generated by $\mathcal{D} = X$, we can extend σ to a mapping from G to $\text{Aut}(X)$. If G is indecomposable, $\sigma(G)$ acts on $X = \mathcal{D}$ transitively. For, if $x \sim y$ with $x, y \in X$, then there exists $z \in X$ such that $x \sim z \sim y$, which implies $\sigma_x(y) = \sigma_y(x) = z$. Thus x and y are conjugate under the action of $\sigma(G)$. Repeating this, we can establish the transitivity.

Given an (indecomposable) 3-transposition group $(G, \mathcal{D})$, one can consider its inductive structure as follows. Let $D^{(1)}$ be the set of elements of $\mathcal{D}$ which commute with a fixed element of $\mathcal{D}$ and let $D^{(2)}$ be that with two fixed non-commuting elements of $\mathcal{D}$. We set $G^{(1)} = \langle D^{(1)} \rangle$ and $G^{(2)} = \langle D^{(2)} \rangle$. The group structure $(G^{(i)}, D^{(i)})$ is called the inductive structure of $(G, \mathcal{D})$.

Proposition 6.3 *Let $(G, \mathcal{D})$ be a centerfree indecomposable 3-transposition group of symplectic type and $(X, \mathcal{L})$ the associated Fischer space. Suppose that $X_{x,y} \neq \emptyset$ for any distinct $x, y \in X$. If $C_{\text{Aut}(X)}(\sigma_x)|_{X_x} = C_G(\sigma_x)|_{X_x}$ for any $x \in X$, then $\text{Aut}(X) = G$.*

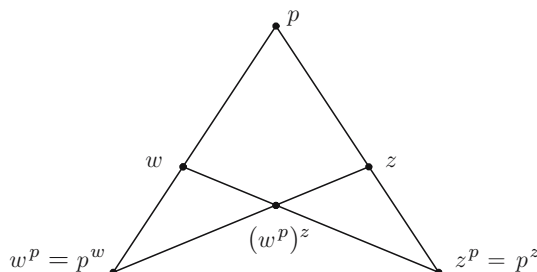
Proof Let $g \in \text{Aut}(X)$. Take any $x \in X = \mathcal{D}$. By the transitivity, there exists $\rho_1 \in G$ such that $g(x) = \rho_1(x)$. Hence $\rho_1^{-1}g|_{X_x} \in C_{\text{Aut}(X)}(\sigma_x)|_{X_x} = C_G(\sigma_x)|_{X_x}$. Then there exists $\rho_2 \in G$ such that $\rho_2^{-1}\rho_1^{-1}g = \text{id}$ on $X_x \cup \{x\}$. Take $y \in X$ such that $y \sim x$. Set $\rho = \rho_1\rho_2$. We show that $\{x, y, y^x\} = X_{x,y}^\perp = \{z \in X \mid z \perp X_{x,y}\}$, from which we deduce that $\rho^{-1}g$ stabilizes the set $\{x, y, y^x\}$. Assume that there exists $z \in X_{x,y}^\perp \setminus \{x, y, y^x\}$. We may assume that $z \sim x$. Since $(X, \mathcal{L})$ is a Fischer space of symplectic type, the subspace $\langle x, y, z \rangle$ generated by x, y and z is isomorphic to a dual affine plane of order 2 and the configuration in the Fischer space is as in Fig. 1:

Fig. 1 Dual affine plane of order 2



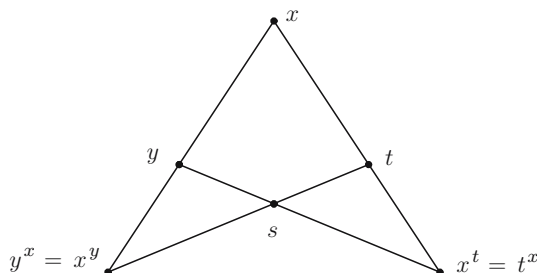
X is indecomposable and $X_{x,y} \neq \emptyset$, there exists $w \in X_{x,y}$ such that w is collinear to a point p in $\langle x, y, z \rangle$. By Fig. 1, $p = z, x^z$ or $(y^x)^z$. Nevertheless, $z \in X_{x,y}^\perp$. Hence we have $p = x^z$ or $p = (y^x)^z$. In either case, $p \sim z$ and $z^p = x$ or y^x . Now consider the subspace generated by w, z and p . Then we have the configuration as in Fig. 2:

Fig. 2 Fisher subspace generated by $\{w, z, p\}$



In this case, $w \sim z^p \in \{x, y^x\}$ which contradicts the fact that $w \in X_{x,y}$. Hence $X_{x,y}^\perp = \{x, y, y^x\}$. Thus by replacing ρ by $\rho\sigma_x$ if needed, we may assume that $\rho^{-1}g(y) = y$. Then $\rho^{-1}g$ acts as identity on $X_x \cup \{x, y, y^x\}$. Now we claim that $\rho^{-1}g = \text{id}$ on X . Take any $t \in X$ such that $t \notin X_x \cup \{x, y, y^x\}$. Then $\{x, y, y^x\}$ and $\{x, t, x^t\}$ are two distinct lines. Since $(X, \mathcal{L})$ is of symplectic type, the Fischer subspace generated by $\{x, y, t\}$ has the configuration as in Fig. 3:

Fig. 3 Fisher subspace generated by $\{x, y, t\}$



Hence, there exists $s \in X_x \cap \langle x, y, t \rangle$ such that $\langle x, y, t \rangle = \langle x, y, s \rangle \simeq S_4$ and $t \in \langle x, y, s \rangle$. Since $\rho^{-1}g$ acts trivially on $\{x, y, s\}$, it acts trivially on t , also. Thus $G = \text{Aut}(X)$. $\square$

6.2 Automorphism group of a commutant subalgebra

We consider a correspondence between an inductive structure of a 3-transposition group and a commutant subalgebra structure of a VOA on which the group acts. We study an inductive structure $O_{10}^+(2)^{(2)} \simeq O_8^-(2)$. It is shown in [13,33] that $\text{Aut}(V_{\sqrt{2}E_8}^+) \simeq O_{10}^+(2)$. By this inductive structure, we can find a commutant subalgebra of $V_{\sqrt{2}E_8}^+$ on which $O_8^-(2)$ acts. We will show that in this case the inductive structure completely determines the full automorphism group of the commutant subalgebra.

Take a root $\alpha_0 \in \Phi(E_8)$ and set $L_1 := \mathbb{Z}\alpha_0$ and $L_2 := (\mathbb{Z}\alpha_0)^\perp = \{\beta \in E_8 \mid \langle \alpha_0, \beta \rangle = 0\}$. Then L_1 and L_2 are sublattices of E_8 isometric to A_1 and E_7 , respectively. Let U be the subalgebra of $V_{\sqrt{2}E_8}^+$ generated by $\tilde{\omega}$ and $\varphi_{\alpha_0}\tilde{\omega}$. It is shown in [19,28] that U is isomorphic to a simple VOA of the form

$$L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right).$$

By a direct computation, one has

$$\tilde{\omega}_{(1)}\varphi_{\alpha_0}\tilde{\omega} = \frac{1}{4}(\tilde{\omega} + \varphi_{\alpha_0}\tilde{\omega} - w^+(\alpha_0))$$

so that $\sigma_{\tilde{\omega}}\varphi_{\alpha_0}\tilde{\omega} = \sigma_{\varphi_{\alpha_0}\tilde{\omega}}\tilde{\omega} = w^+(\alpha_0)$ (cf. Proposition 3.4). Therefore, U contains exactly three Ising vectors $\tilde{\omega}$, $\varphi_{\alpha_0}\tilde{\omega}$ and $w^+(\alpha_0)$. Set

$$U^c = \text{Com}_{V_{\sqrt{2}E_8}}(U).$$

We shall show that U^c is again generated by a set of Ising vectors and its full automorphism group $\text{Aut}(U^c)$ is isomorphic to $O_8^-(2)$.

Proposition 6.4 *There are exactly 136 Ising vectors in the commutant subalgebra U^c and they are mutually conjugate under the action of $\text{Aut}(U^c)$.*

Proof Since U is generated by $\tilde{\omega}$ and $\varphi_{\alpha_0}\tilde{\omega}$, we have

$$U^c = \text{Com}_{V_{\sqrt{2}E_8}^+}(\text{Vir}(\tilde{\omega})) \cap \text{Com}_{V_{\sqrt{2}E_8}^+}(\text{Vir}(\varphi_{\alpha_0}\tilde{\omega})).$$

By Proposition 4.3 and Lemma 4.8, if e is an Ising vector in U^c , then either

- (1) $e = w^-(\beta)$ for some β with $\langle \alpha_0, \beta \rangle = 0, \pm 2$ or
 (2) $e = \varphi_x(\tilde{\omega})$, $x + 2E_8$ is represented by a norm 4 vector and $\langle \alpha_0, x \rangle \equiv 1 \pmod{2}$.

Note that the lattice $L_2 = (\mathbb{Z}\alpha_0)^\perp$ is isomorphic to E_7 , which has 63 positive roots. Moreover, $\langle \alpha_0, \beta \rangle = \pm 2$ if and only if $\beta = \pm \alpha_0$. Hence, there are exactly $1 + 63 = 64$ Ising vectors of the form $w^-(\beta)$ in the case (1).

For the case (2), there are exactly 128 cosets $x + 2E_8$ of $E_8/2E_8$ such that $\langle \alpha_0, x \rangle = 1 \pmod{2}$. Among them, $120 - 64 = 56$ classes are represented by roots. Hence there are 72 Ising vectors of the form $\varphi_x(\tilde{\omega})$ in U^c . Therefore, there are totally $64 + 72 = 136$ Ising vectors in U^c .

Next we shall show all Ising vectors are mutually conjugate in U^c . First we shall note that the Weyl group $W(L_2) \simeq W(E_7)$ is naturally a subgroup of $\text{Aut}(U^c)$ since both $\tilde{\omega}$ and $w^+(\alpha_0)$ are fixed by $W(L_2)$. Hence, it is clear that $w^-(\beta)$ and $w^-(\gamma)$ are conjugate if $\beta, \gamma \in L_2$. Let $e = \varphi_x(\tilde{\omega})$ be an Ising vector of U^c . Then $x + 2E_8$ is represented by a norm 4 vector and $\langle \alpha_0, x \rangle \equiv 1 \pmod{2}$. Since E_8 is generated by its roots, we may set $x = \beta_1 + \beta_2$ such that β_1 and β_2 are roots in E_8 and $\langle \beta_1, \beta_2 \rangle = 0$, $\langle \alpha_0, \beta_1 \rangle = 1$ and $\langle \alpha_0, \beta_2 \rangle = 0$. In this case, $\beta_2 \in (\mathbb{Z}\alpha_0 + \mathbb{Z}\beta_1)^\perp \simeq E_6$ and hence there exists a root $\gamma \in (\mathbb{Z}\alpha_0 + \mathbb{Z}\beta_1)^\perp$ such that $\langle \beta_1, \gamma \rangle = \langle \alpha_0, \gamma \rangle = 0$ and $\langle \beta_2, \gamma \rangle = 1$. Therefore, we have $\langle \varphi_x(\tilde{\omega}), w^-(\gamma) \rangle = 1/32$ and thus $\varphi_x(\tilde{\omega})$ is conjugate to $w^-(\gamma)$ by (2) of Proposition 3.4.

Finally, for any $e = \varphi_x(\tilde{\omega}) \in U^c$, we have $\langle \alpha_0, x \rangle \equiv 1 \pmod{2}$. Hence, we have $\langle \varphi_x(\tilde{\omega}), w^-(\alpha_0) \rangle = 1/32$ and $\varphi_x(\tilde{\omega})$ is conjugate to $w^-(\alpha_0)$ also. $\square$

Lemma 6.5 U^c is generated by its Ising vectors.

Proof Denote

$$L = \sqrt{2}(L_1 \oplus L_2) \simeq \sqrt{2}A_1 \oplus \sqrt{2}E_7.$$

Since $|\sqrt{2}E_8/L| = 2$, there is $\gamma \in \sqrt{2}E_8$ such that $\sqrt{2}E_8 = L \cup (\gamma + L)$ and hence

$$V_{\sqrt{2}E_8} = V_L \oplus V_{\gamma+L} \quad \text{and} \quad V_{\sqrt{2}E_8}^+ = V_L^+ \oplus V_{\gamma+L}^+.$$

Note that the quotient group structure $\sqrt{2}E_8/L$ induces an automorphism $\rho \in \text{Aut}(V_{\sqrt{2}E_8}^+)$ such that $\rho|_{V_L^+} = 1$ and $\rho|_{V_{\gamma+L}^+} = -1$.

By definition (cf. (2.3)), it is easy to show

$$\tilde{\omega}_{L_2} (= \tilde{\omega}_{E_7}) = \frac{4}{5}(\tilde{\omega} + \varphi_{\alpha_0}\tilde{\omega}) - \frac{1}{5}w^+(\alpha_0).$$

Therefore, $\tilde{\omega}_{L_2} \in U$ and the Virasoro element of U is an orthogonal sum of $w^+(\alpha_0)$ and $\tilde{\omega}_{L_2}$. Hence the commutant subalgebra U^c can be defined as follows.

$$U^c = \{v \in V_{\sqrt{2}E_8}^+ \mid (\tilde{\omega}_{L_2})_{(1)}v = w^+(\alpha_0)_{(1)}v = 0\}.$$

Let

$$\begin{aligned} M^0 &= U^c \cap V_L^+ = \{v \in V_L^+ \mid (\tilde{\omega}_{L_2})_{(1)}v = w^+(\alpha_0)_{(1)}v = 0\}, \\ M^1 &= U^c \cap V_{\gamma+L}^+ = \{v \in V_{\gamma+L}^+ \mid (\tilde{\omega}_{L_2})_{(1)}v = w^+(\alpha_0)_{(1)}v = 0\}. \end{aligned}$$

Then we have $U^c = M^0 \oplus M^1$. Moreover, the automorphism ρ induces a natural action on U^c such that $\rho|_{M^0} = 1$ and $\rho|_{M^1} = -1$. Note that

$$M^0 \simeq L(1/2, 0) \otimes M_{E_7}$$

and M^1 is an irreducible M^0 -module in this case (cf. [8]).

Now by Lemma 4.8 and Proposition 5.1, we know that M_{E_7} is generated by the Ising vectors of the form $w^-(\beta)$, $\beta \in L_2 \simeq E_7$. Hence, $M^0 \simeq L(\frac{1}{2}, 0) \otimes M_{E_7}$ is generated by $w^-(\alpha_0)$ and $\{w^-(\beta) \mid \beta \in L_2 \simeq E_7\}$.

Let W be the sub-VOA generated by the set of all Ising vectors of U^c . Then $M^0 \subset W$. Since W also contains Ising vectors of the form $\varphi_x(\tilde{\omega})$ which is not contained in M^0 (cf. Lemma 4.8), we know that $W \neq M^0$. Hence, $W = M^0 \oplus M^1 = U^c$ as desired. $\square$

Theorem 6.6 $\text{Aut}(U^c) \simeq \text{O}_8^-(2)$.

Proof Let E be the set of Ising vectors of U^c and G the 3-transposition subgroup of $\text{Aut}(U^c)$ generated by involutions $\{\sigma_e \mid e \in E\}$. It is shown in [25] that $G \simeq \text{O}_8^-(2)$. By the proof of the previous lemma, we have

$$\tilde{\omega}_{L_2} = \frac{4}{5}(\tilde{\omega} + \varphi_{\alpha_0}\tilde{\omega}) - \frac{1}{5}w^+(\alpha_0) \in U$$

and $\text{Com}_{U^c}(\text{Vir}(w^-(\alpha_0))) \simeq M_{E_7}$. By Theorem 5.4, $\text{Aut}(M_{E_7})$ is isomorphic to $\text{Sp}_6(2)$ which is generated by σ -type involutions associated to Ising vectors of M_{E_7} . Since the 3-transposition subgroup of G generated by $\sigma(E)_{\sigma_{w^-(\alpha)}} = \{\sigma_{w^-(\beta)} \mid \beta \in L_2\}$ is isomorphic to $\text{Sp}_6(2)$, we can apply Proposition 6.3 to G and we conclude that $\text{Aut}(U^c) = G \simeq \text{O}_8^-(2)$ as U^c is generated by E . $\square$

7 Decomposition of M_R

Next, we will complete the proof of Proposition 5.1. Since it requires the notion of W -algebras, we will first review some basic facts about W -algebras.

7.1 Modules over W -algebra

Let Λ_0 and Λ_1 be the fundamental weights of the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$. For positive integers ℓ, j with $0 \leq j \leq \ell$, consider the irreducible highest weight module $\mathcal{L}(\ell, j)$ over $\hat{\mathfrak{sl}}_2(\mathbb{C})$ with highest weight $(\ell - j)\Lambda_0 + j\Lambda_1$. It is well known that $\mathcal{L}(\ell, 0)$ forms a simple VOA and the integrable $\hat{\mathfrak{sl}}_2(\mathbb{C})$ -modules $\mathcal{L}(\ell, j)$, $0 \leq j \leq \ell$, provide all the inequivalent irreducible $\mathcal{L}(\ell, 0)$ -modules (cf. [11]).

Now let $A_1^{\oplus \ell} = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_\ell$ be an even lattice with $\langle \epsilon_i, \epsilon_j \rangle = 2\delta_{ij}$ and $V_{A_1^{\oplus \ell}}$ the lattice VOA associated with $A_1^{\oplus \ell}$. Then $V_{A_1^{\oplus \ell}} \simeq (V_{A_1})^{\otimes \ell} \simeq \mathcal{L}(1, 0)^{\otimes \ell}$. Recall that the weight one subspace of $V_{A_1^{\oplus \ell}}$ forms a Lie algebra by the Lie bracket $[x, y] := x_{(0)}y$ for $x, y \in (V_{A_1^{\oplus \ell}})_1$. Set $H^{(\ell)} = (\epsilon_1 + \cdots + \epsilon_\ell)_{(-1)}\mathbf{1}$, $E^{(\ell)} = e^{\epsilon_1} + \cdots + e^{\epsilon_\ell}$ and $F^{(\ell)} = e^{-\epsilon_1} + \cdots + e^{-\epsilon_\ell}$. Then the subspace $\mathbb{C}H^{(\ell)} + \mathbb{C}E^{(\ell)} + \mathbb{C}F^{(\ell)}$ of the weight one subspace of $V_{A_1^{\oplus \ell}}$ forms a simple Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and the sub-VOA generated by $\{H^{(\ell)}, E^{(\ell)}, F^{(\ell)}\}$ is isomorphic to level ℓ affine VOA $\mathcal{L}(\ell, 0)$ (cf. [6]).

Let $\gamma = \epsilon_1 + \cdots + \epsilon_\ell \in A_1^{\oplus \ell}$. Then $\gamma_{(-1)}\mathbf{1} = H^{(\ell)}$ and it is easy to verify that

$$e^\gamma = \frac{1}{\ell!}(E_{(-1)}^{(\ell)})^{\ell-1}E^{(\ell)}.$$

Thus $\mathcal{L}(\ell, 0)$ contains a subalgebra isomorphic to the lattice VOA $V_{\mathbb{Z}\gamma}$. We consider the following commutant subalgebra:

$$W_\ell := \text{Com}_{\mathcal{L}(\ell, 0)}(V_{\mathbb{Z}\gamma}). \quad (7.1)$$

Note that the central charge of W_ℓ is equal to $3\ell/(\ell + 2) - 1 = 2(\ell - 1)/(\ell + 2)$. It is clear that $\mathcal{L}(\ell, 0)$ contains a full sub-VOA isomorphic to $V_{\mathbb{Z}\gamma} \otimes W_\ell$ so that we can regard $\mathcal{L}(\ell, j)$ as a $V_{\mathbb{Z}\gamma} \otimes W_\ell$ -module. Following [6, 24], we introduce the following spaces:

$$W_\ell(j, k) := \text{Hom}_{V_{\mathbb{Z}\gamma}}(V_{(k/2\ell)\gamma + \mathbb{Z}\gamma}, \mathcal{L}(\ell, j)), \quad (7.2)$$

where $0 \leq j \leq \ell$ and $0 \leq k < 2\ell$. Then $W_\ell(j, k)$ denotes the space of multiplicity of $V_{(k/2\ell)\gamma + \mathbb{Z}\gamma}$ in $\mathcal{L}(\ell, j)$. Thus, viewing $\mathcal{L}(\ell, j)$ as a $V_{\mathbb{Z}\gamma} \otimes W_\ell$ -module, we have the following decomposition:

$$\mathcal{L}(\ell, j) = \bigoplus_{k=0}^{2\ell-1} V_{(k/2\ell)\gamma + \mathbb{Z}\gamma} \otimes W_\ell(j, k).$$

It is shown in [6] that $W_\ell(j, k) = 0$ if $j + k \equiv 1 \pmod{2}$, and so

$$\mathcal{L}(\ell, j) = \begin{cases} \bigoplus_{k=0}^{\ell-1} V_{(k/\ell)\gamma + \mathbb{Z}\gamma} \otimes W_\ell(j, 2k), & \text{if } j \text{ is even,} \\ \bigoplus_{k=0}^{\ell-1} V_{((2k+1)/2\ell)\gamma + \mathbb{Z}\gamma} \otimes W_\ell(j, 2k+1), & \text{if } j \text{ is odd.} \end{cases} \quad (7.3)$$

The following basic fact is due to [6] (see also [24]).

Proposition 7.1 ([6, 24])

- (1) All $W_\ell(j, k)$, $0 \leq j \leq \ell$, $0 \leq k \leq 2\ell - 1$, $j \equiv k \pmod{2}$, are irreducible W_ℓ -modules.
- (2) As W_ℓ -modules, $W_\ell(j_1, k_1) \simeq W_\ell(j_2, k_2)$ if $j_1 + j_2 = \ell$ and $k_2 \equiv k_1 + \ell \pmod{2\ell}$.

We will use W_ℓ -modules $W_\ell(j, k)$ to study M_{E_6} and M_{E_7} . In [23], they recursively computed the vacuum characters of these modules. Below we shall use the vacuum characters of W_ℓ -modules obtained in (loc. cit.) without any comments.

7.2 $V_{\sqrt{2}A_N}$ and W_{N+1} -algebra

Let us consider a lattice VOA $V_{\sqrt{2}A_N}$. Let s_{A_N} and $\tilde{\omega}_{A_N}$ be conformal vectors of $V_{\sqrt{2}A_N}$ defined as in (2.2) and (2.3), respectively. Then one has an orthogonal decomposition $\omega = s_{A_N} + \tilde{\omega}_{A_N}$ of the Virasoro vector ω of $V_{\sqrt{2}A_N}$. We use the following result established in [20].

Lemma 7.2 ([20]) *There is a VOA-isomorphism $\text{Com}_{V_{\sqrt{2}A_N}}(M_{A_N}) \simeq W_{N+1}$.*

Due to the lemma above, we shall identify the commutant subalgebra $\text{Com}_{V_{\sqrt{2}A_N}}(M_{A_N})$ with W_{N+1} . By the orthogonality, $(W_{N+1}, \tilde{\omega}_{A_N})$ forms a sub-VOA of $V_{\sqrt{2}A_N}$. Therefore, $V_{\sqrt{2}A_N}$ contains a full subalgebra isomorphic to $M_{A_N} \otimes W_{N+1}$. It is shown

in [7] that M_{A_N} contains a full sub-VOA isomorphic to $L(c_1, 0) \otimes \cdots \otimes L(c_N, 0)$, where c_m denotes the central charge of the unitary series of the Virasoro algebra:

$$c_m := 1 - \frac{6}{(m+2)(m+3)}, \quad m = 1, 2, \dots \quad (7.4)$$

By [35], the irreducible modules over $L(c_m, 0)$ are given by the irreducible highest weight modules $L(c_m, h_{r,s}^m)$ whose highest weights are parameterized as follows.

$$h_{r,s}^m := \frac{\{r(m+3) - s(m+2)\}^2 - 1}{4(m+2)(m+3)}, \quad 1 \leq r \leq m+1, \quad 1 \leq s \leq m+2. \quad (7.5)$$

In [20], the following decomposition of $V_{\sqrt{2}A_N}$ as an $L(c_1, 0) \otimes \cdots \otimes L(c_N, 0) \otimes W_{N+1}$ -module is obtained.

$$V_{\sqrt{2}A_N} = \bigoplus_{\substack{0 \leq k_j \leq j+1 \\ j=0, \dots, N \\ k_j \equiv 0 \pmod{2}}} L(c_1, h_{k_0+1, k_1+1}^1) \otimes \cdots \otimes L(c_N, h_{k_{N-1}+1, k_N+1}^N) \otimes W_{N+1}(k_N, 0). \quad (7.6)$$

To describe $V_{\sqrt{2}A_N}$ as an $M_{A_N} \otimes W_{N+1}$ -module, we introduce the following notation.

$$M_{A_N}(2s) := \bigoplus_{\substack{0 \leq k_j \leq j+1 \\ j=0, \dots, N-1 \\ k_j \equiv 0 \pmod{2}}} L(c_1, h_{k_0+1, k_1+1}^1) \otimes \cdots \otimes L(c_N, h_{k_{N-1}+1, 2s+1}^N). \quad (7.7)$$

By (7.6), we have

$$V_{\sqrt{2}A_N} = \bigoplus_{0 \leq 2s \leq N+1} M_{A_N}(2s) \otimes W_{N+1}(2s, 0) \quad (7.8)$$

and it is shown in [18] that $M_{A_N}(2s)$, $0 \leq 2s \leq N+1$, are inequivalent irreducible M_{A_N} -modules. By construction, the vacuum characters of $M_{A_N}(2s)$ are obvious.

7.3 Decomposition of M_{E_7}

Let us recall that the root lattice of type E_7 can be written as

$$E_7 = \left\{ (x_1, \dots, x_8) \in \mathbb{Q}^8 \mid \begin{array}{l} \text{all } x_i \text{ are in } \mathbb{Z} \text{ or all } x_i \text{ are in } \frac{1}{2} + \mathbb{Z}, \\ x_1 + x_2 + \cdots + x_7 + x_8 = 0 \end{array} \right\}.$$

Let ϵ_i be the vector of $\mathbb{Q}^8$ such that the i -th entry is 1 and all the other entries are zero, and set

$$N := \text{Span}_{\mathbb{Z}}\{-\epsilon_1 + \epsilon_2, \dots, -\epsilon_7 + \epsilon_8\}.$$

Then N is a root lattice of type A_7 . Let $\xi = (1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2) \in \mathbb{Q}^8$. Then the root lattice of type E_7 can be written as $E_7 = N \cup (\xi + N)$.

Let s_{A_7} and $\tilde{\omega}_{A_7} = \omega - s_{A_7}$ be conformal vectors defined as in (2.2) and (2.3), respectively. Then W_8 is isomorphic to the parafermion algebra of central charge $7/5$ (cf. [20, 37]). As we have seen, $V_{\sqrt{2}N}$ contains a full sub-VOA isomorphic to $M_{A_7} \otimes W_8$. The following decomposition is obtained in [18, 20].

Lemma 7.3 ([18, 20]) *As a module over $M_{A_7} \otimes W_8$,*

$$V_{\sqrt{2}\xi + \sqrt{2}N} \simeq \bigoplus_{0 \leq 2s \leq 8} M_{A_7}(2s) \otimes W_8(2s, 8).$$

Hence, we have

$$V_{\sqrt{2}E_7} = V_{\sqrt{2}N} \oplus V_{\sqrt{2}\xi + \sqrt{2}N} \simeq \bigoplus_{0 \leq 2s \leq 8} M_{A_7}(2s) \otimes (W_8(2s, 0) \oplus W_8(2s, 8)).$$

Let $\tilde{\omega}'_{E_7} = \tilde{\omega}_{A_7} - \tilde{\omega}_{E_7}$. Then $\tilde{\omega}_{E_7}$ and $\tilde{\omega}'_{E_7}$ are mutually orthogonal conformal vectors with central charge $7/10$. Denote

$$U := \text{Com}_{V_{\sqrt{2}E_7}}(\text{Vir}(s_{A_7})) \simeq W_8(0, 0) \oplus W_8(0, 8).$$

Then U contains $\tilde{\omega}_{E_7}$ and $\tilde{\omega}'_{E_7}$ so that $\text{Vir}(\tilde{\omega}'_{E_7}) \otimes \text{Vir}(\tilde{\omega}_{E_7})$ is a full sub-VOA of U isomorphic to $L(7/10, 0) \otimes L(7/10, 0)$. By the vacuum characters, it is easy to show that

$$U \simeq L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right)$$

as a $\text{Vir}(\tilde{\omega}'_{E_7}) \otimes \text{Vir}(\tilde{\omega}_{E_7})$ -module. Moreover, we have

Lemma 7.4 *The lattice VOA $V_{\sqrt{2}E_7}$ can be decomposed as follows:*

$$V_{\sqrt{2}E_7} \simeq \bigoplus_{0 \leq 2s \leq 8} M_{A_7}(2s) \otimes U(2s),$$

where

$$\begin{aligned} U(0) &\simeq U(8) \simeq L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right), \\ U(2) &\simeq U(6) \simeq L\left(\frac{7}{10}, \frac{3}{5}\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \oplus L\left(\frac{7}{10}, \frac{1}{10}\right) \otimes L\left(\frac{7}{10}, \frac{1}{10}\right), \\ U(4) &\simeq L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \oplus L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, \frac{1}{10}\right) \\ &\quad \oplus L\left(\frac{7}{10}, \frac{3}{5}\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{7}{10}, \frac{1}{10}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right). \end{aligned}$$

Since $\omega = s_{A_7} + \tilde{\omega}'_{E_7} + \tilde{\omega}_{E_7}$ is an orthogonal sum, we have the following isomorphism as $M_{A_7} \otimes \text{Vir}(\tilde{\omega}'_{E_7})$ -modules:

$$M_{E_7} \simeq M_{A_7}(0) \otimes L\left(\frac{7}{10}, 0\right) \oplus M_{A_7}(4) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \oplus M_{A_7}(8) \otimes L\left(\frac{7}{10}, 0\right).$$

Proposition 7.5 *M_{E_7} is generated by its weight 2 subspace.*

Proof It is shown in Proposition 5.6 of [18] that M_{A_N} is generated by its weight two subspace as a VOA. Thus so is $M_{A_7}(0) \otimes L(7/10, 0)$. Recall that

$$M_{A_7}(2s) \simeq \bigoplus_{\substack{0 \leq k_j \leq j+1, \\ j=0, \dots, 7 \\ k_j \equiv 0 \pmod{2}}} L\left(c_1, h_{k_0+1, k_1+1}^1\right) \otimes \cdots \otimes L\left(c_7, h_{k_6+1, 2s+1}^7\right).$$

Take $(k_0 + 1, k_1 + 1, \dots, k_6 + 1, 2s + 1) = (1, 3, 3, 5, 5, 7, 7, 9)$. Then we obtain a highest weight vector of weight

$$\left(\frac{1}{2}, \frac{1}{10}, \frac{2}{5}, \frac{1}{7}, \frac{5}{14}, \frac{1}{6}, \frac{1}{3}\right)$$

in $M_{A_7}(8)$. Similarly, by taking $(k_0 + 1, k_1 + 1, \dots, k_6 + 1, 2s + 1) = (1, 1, 1, 1, 1, 3, 5, 5)$, $M_{A_7}(4)$ contains a highest weight vector of weight

$$\left(0, 0, 0, 0, \frac{3}{4}, \frac{7}{12}, \frac{1}{15}\right).$$

Therefore, both $M_{A_7}(4) \otimes L(\frac{7}{10}, \frac{3}{5})$ and $M_{A_7}(8) \otimes L(\frac{7}{10}, 0)$ contain weight 2 elements. Since $M_{A_7}(4) \otimes L(\frac{7}{10}, \frac{3}{5})$ and $M_{A_7}(8) \otimes L(\frac{7}{10}, 0)$ are irreducible $M_{A_7} \otimes L(\frac{7}{10}, 0)$ -modules (cf. [18]), M_{E_7} is generated by its weight 2 subspace. $\square$

7.4 Decomposition of M_{E_6}

Let us recall that

$$E_6 = \left\{ (x_1, \dots, x_8) \in \mathbb{Q}^8 \mid \begin{array}{l} \text{all } x_i \text{ are in } \mathbb{Z} \text{ or all } x_i \text{ are in } \frac{1}{2} + \mathbb{Z}, \\ \text{and } x_1 + x_8 = x_2 + \dots + x_7 = 0 \end{array} \right\}.$$

Define

$$\begin{aligned} L_1 &= \left\{ (0; x_2, \dots, x_7; 0) \in \mathbb{Z}^8 \mid x_2 + \dots + x_7 = 0 \right\}, \\ L_2 &= \left\{ (x_1; 0, \dots, 0; x_8) \in \mathbb{Z}^8 \mid x_1 + x_8 = 0 \right\}. \end{aligned}$$

Then $L_1 \simeq A_5$, $L_2 \simeq A_1$ and it gives an embedding of $A_5 \oplus A_1$ into E_6 . Set

$$\xi = \left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}\right) \quad \text{and} \quad L = L_1 \oplus L_2.$$

Then we have $E_6 = L \cup (\xi + L)$ and $V_{\sqrt{2}E_6} = V_{\sqrt{2}L} \oplus V_{\sqrt{2}\xi + \sqrt{2}L}$. Note that

$$V_{\sqrt{2}L} \simeq V_{\sqrt{2}A_5} \otimes V_{\sqrt{2}A_1} \quad \text{and} \quad V_{\sqrt{2}\xi + \sqrt{2}L} \simeq V_{\sqrt{2}\xi_1 + \sqrt{2}A_5} \otimes V_{\sqrt{2}\xi_2 + \sqrt{2}A_1},$$

where we have set $\xi_1 = (0; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 0)$ and $\xi_2 = (\frac{1}{2}; 0, 0, 0, 0, 0, 0; -\frac{1}{2})$.

Define $s_{A_5}, \tilde{\omega}_{A_5} \in V_{L_1}$ and $w^\pm(2\xi_2) \in V_{L_2}$ as in (2.2), (2.3) and (4.4), respectively, and we identify (M_{A_5}, s_{A_5}) as a sub-VOA of $V_{L_1} \simeq V_{\sqrt{2}A_5}$. Set

$$\omega^1 := \tilde{\omega}_{A_5} + w^+(2\xi_2) - \tilde{\omega}_{E_6} \in V_{\sqrt{2}E_6} \quad \text{and} \quad \omega^2 := w^-(2\xi_2) \in V_{L_2}. \quad (7.9)$$

Then ω^1 and ω^2 are conformal vectors of $V_{\sqrt{2}E_6}$ with central charges $25/28$ and $1/2$, respectively. We also note that $\text{Vir}(\omega^1) \simeq L(\frac{25}{28}, 0)$ and $\text{Vir}(\omega^2) \simeq L(\frac{1}{2}, 0)$. One can directly check that $\omega = s_{A_5} + \omega^1 + \omega^2 + \tilde{\omega}_{E_6}$ is an orthogonal sum (cf. Lemma 2.1). Therefore, $M_{\sqrt{2}E_6}$ contains a full sub-VOA $M_{A_5} \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \simeq M_{A_5} \otimes L(\frac{25}{28}, 0) \otimes L(\frac{1}{2}, 0)$. By a similar method as in [18, 20], one can establish the following.

Lemma 7.6 *As a module over $M_{A_5} \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$,*

$$\begin{aligned} M_{E_6} \simeq & M_{A_5}(0) \otimes \left\{ L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \right\} \\ & \oplus M_{A_5}(2) \otimes \left\{ L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \right\} \\ & \oplus M_{A_5}(4) \otimes \left\{ L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \right\} \\ & \oplus M_{A_5}(6) \otimes \left\{ L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \right\}. \end{aligned}$$

By (7.7), M_{A_5} contains a sub-VOA isomorphic to M_{A_4} such that $\text{Com}_{M_{A_5}}(M_{A_4}) \simeq L(\frac{25}{28}, 0)$. Therefore, M_{A_5} has a full sub-VOA isomorphic to $M_{A_4} \otimes L(\frac{25}{28}, 0)$ and we have the following decomposition:

$$M_{A_5}(s) = \bigoplus_{m=0,2,4} M_{A_4}(m) \otimes L\left(\frac{25}{28}, h_{m+1,s+1}^5\right). \quad (7.10)$$

Denote the conformal vectors of M_{A_4} and $\text{Com}_{M_{A_5}}(M_{A_4})$ by u and v , respectively. Then $\tilde{\omega}_{A_5} = u + v$ is an orthogonal sum and we have the following sequence of the full sub-VOAs of $V_{\sqrt{2}E_6}$:

$$M_{A_4} \otimes \text{Vir}(v) \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \subset M_{A_5} \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \subset M_{E_6}.$$

Thus $\text{Com}_{M_{E_6}}(M_{A_4})$ contains a full sub-VOA $\text{Vir}(v) \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$ isomorphic to $L(\frac{25}{28}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{1}{2}, 0)$. By Lemma 7.6 and (7.10), we have

Lemma 7.7 *As a module over $\text{Vir}(v) \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$,*

$$\begin{aligned} & \text{Com}_{M_{E_6}}(M_{A_4}) \\ & \simeq L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{1}{2}, 0\right) \\ & \quad \oplus L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{1}{2}, 0\right) \\ & \quad \oplus L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \oplus L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{25}{28}, \frac{3}{4}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \\ & \quad \oplus L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{25}{28}, \frac{13}{4}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \oplus L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

The key observation is that $\text{Com}_{M_{E_6}}(M_{A_4})$ is isomorphic to a $\mathbb{Z}_2$ -orbifold subalgebra of the 5A-algebra considered in [23]. We refer the proof of the following fact to [23].

Lemma 7.8 ([23]) *$\text{Com}_{M_{E_6}}(M_{A_4})$ is generated by its weight two subspace.*

Now we can show the following.

Proposition 7.9 *M_{E_6} is generated by its weight two subspace.*

Proof Let U be the sub-VOA of M_{E_6} generated by the weight two subspace of M_{E_6} . Since M_{A_5} is generated by its weight two subspace (cf. Proposition 5.6 of [18]), U contains M_{A_5} . It is also shown in [18] that all $M_{A_5}(s)$, $s = 0, 2, 4, 6$, are irreducible M_{A_5} -submodules of M_{E_6} . On the other hand, thanks to Lemma 7.8, U also contains all the highest weight vectors for $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$ which appear in the decomposition in Lemma 7.6. Therefore, U contains all the components given in Lemma 7.6 and hence $U = M_{E_6}$. $\square$

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