Compositio Math. 142 (2006) 616–630
doi:10.1112/S0010437X05001788

On vector bundles destabilized by Frobenius pull-back

Kirti Joshi, S. Ramanan, Eugene Z. Xia and Jiu-Kang Yu

Abstract

Let X be a smooth projective curve of genus $g > 1$ over an algebraically closed field of positive characteristic. This paper is a study of a natural stratification, defined by the absolute Frobenius morphism of X, on the moduli space of vector bundles. In characteristic two, there is a complete classification of semi-stable bundles of rank 2 which are destabilized by Frobenius pull-back. We also show that these strata are irreducible and obtain their respective dimensions. In particular, the dimension of the locus of bundles of rank two which are destabilized by Frobenius is $3g - 4$. These Frobenius destabilized bundles also exist in characteristics two, three and five with ranks 4, 3 and 5, respectively. Finally, there is a connection between (pre)-opers and Frobenius destabilized bundles. This allows an interpretation of some of the above results in terms of pre-opers and provides a mechanism for constructing Frobenius destabilized bundles in large characteristics.

1. Introduction

Let X be an irreducible smooth projective curve of genus g over an algebraically closed field k of characteristic $p > 0$, and let $F : X \to X$ be the absolute Frobenius morphism on X. It is known that pulling back a stable vector bundle on X by F may destroy stability. One may measure the failure of (semi-)stability by the Harder–Narasimhan polygons of vector bundles.

In more formal language, let $n \geqslant 2$ be an integer, and let $\mathcal{M}$ be the coarse moduli space of stable vector bundles of rank n and a fixed degree on X. Applying a theorem of Shatz to the pull-back by F of the universal bundle (assuming the existence) on $\mathcal{M}$, we see that $\mathcal{M}$ has a canonical stratification by Harder–Narasimhan polygons [LeP97]. We call this the *Frobenius stratification.* This interesting extra structure on $\mathcal{M}$ is a feature of characteristic $p > 0$. However, very little is known about the strata of the Frobenius stratification. Scattered constructions of points outside of the largest (semi-stable) stratum can be found in [Gie73, RR84, Ray82]. Complete classification of such points is only known when $p = 2$, $n = 2$, and $g = 2$ by [JX00, LP02, LP04] ($g = 2, p = 3$). See also [Oss04] and [Jos03] for results with other values of g, p, n.

Our main result here settles the problem for the case of $p = 2$ and $n = 2$. On any curve X of genus $\geqslant 2$, we provide a complete classification of rank-2 semi-stable vector bundles V with F^*V not semi-stable. This also shows that the bound in [Sun99, Theorem 3.1] is sharp. We also obtain fairly good information about the locus destabilized by Frobenius in the moduli space, including the irreducibility and the dimension of each non-empty Frobenius stratum. In particular, we show that the locus of Frobenius destabilized bundles has dimension $3g - 4$ in the moduli space of semi-stable

Received 13 December 2004, accepted in final form 9 June 2005.
2000 Mathematics Subject Classification 14H60 (primary), 13A35, 14D20 (secondary).
Keywords: Projective curves, characteristic p, Frobenius morphism.
Eugene Z. Xia was partially supported by grants NSC 91-2115-M-006-022 and NSC 92-2115-M-006-008 of the National Science Council of Taiwan. Jiu-Kang Yu was partially by a Sloan Foundation Fellowship and by grant DMS 0100678 from the National Science Foundation.

bundles of rank 2. An interesting consequence of our classification is that high instability of F^*V implies high stability of V.

In addition, we show that the Gunning bundle descends when g is even. If g is odd, then the Gunning bundle twisted by any odd degree line bundle also descends.

We also construct stable bundles that are destabilized by Frobenius in the following situations: (i) $p = 2$ and $n = 4$; (ii) $p = n = 3$; and (iii) $p = n = 5$ and $g \geqslant 3$.

The problem studied here can be cast in the generality of principal G-bundles over X, where G is a connected reductive group over k. More precisely, consider the pull-back by F of the universal object on the moduli stack of semi-stable principal G-bundles on X. Atiyah–Bott's generalization of the Harder–Narasimhan filtration should then give a canonical stratification of the moduli stack ([AB82], see also [Cha00]).

There is a connection between Frobenius destabilized bundles and (pre)-opers. The investigation of this connection is largely inspired by [BD00]. When $p = 0$, all pre-opers are opers. The new phenomena observed here is that, in characteristic $p > 0$, pre-opers need not be opers. Section 5.3 contains many such examples. These pre-opers rise from vector bundles destabilized by Frobenius pull-back.

2. Generalities

2.1 Notation

The following notation are in force throughout this paper unless otherwise specified. Let X be a smooth, projective curve of genus $g \geqslant 2$ over an algebraically closed field k of characteristic $p > 0$. Let Ω^1_X be the sheaf of 1-forms on X and let T_X be the tangent bundle of X.

For any k-scheme Y, the absolute Frobenius $F : Y \to Y$ is defined by the $f \mapsto f^p$ map in $\mathcal{O}_Y$. Notice that F is an $\mathbb{F}_p$-morphism only, not a k-morphism. There is a commutative diagram

$$\begin{tikzcd}
Y \arrow[dr, "F_{\mathrm{r}}"] \arrow[drr, "F"] \arrow[ddr] & & \\
& Y^{(p)} \arrow[r, "\varpi_Y"] \arrow[d] & Y \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "F"] & \operatorname{Spec} k
\end{tikzcd}$$

in which F_{r} is the relative Frobenius morphism, and the square is Cartesian. Since k is perfect, $F : \operatorname{Spec} k \to \operatorname{Spec} k$ and $\varpi_Y : Y^{(p)} \to Y$ are isomorphisms. Most of the results in this article can be formulated using either $F : Y \to Y$ or $F_{\mathrm{r}} : Y^{(p)} \to Y$. Following [JX00] and [Gie75], we choose to work with the former to avoid working with $Y^{(p)}$ all the time. This also makes results such as Lemma 2.8 slightly easier to state. We caution the reader that many papers on this subject work with the relative Frobenius. However, the translation between the different conventions is simple.

For any two k-schemes X and Y, we shorten $Y \times_{\operatorname{Spec} k} X$ to $Y \times X$. Note also that there is an $\mathbb{F}_p$-morphism

$$\varpi_Y \times F : Y^{(p)} \times X \to Y \times X.$$

We shorten ϖ_Y to ϖ when there is no confusion.

Let V be a vector bundle on X and denote by $F^*(V)$ the pull-back of V by F. If $V = L$ is a line bundle, then $F^*(L) = L^{\otimes p}$. We write $V^* = \operatorname{Hom}_{\mathcal{O}_X}(V, \mathcal{O}_X)$ for the dual bundle of V and $\chi(V)$

for the Euler characteristic of V. By Riemann–Roch,

$$\chi(V) = \deg(V) + \operatorname{rank}(V)(1-g).$$

Denote by $\mu(V) = \deg(V)/\operatorname{rank}(V)$ the *slope* of V.

2.2 Stability

A vector bundle V is *stable* (respectively *semi-stable*) if for any non-zero sub-bundle $W \subset V$, $\mu(W) < \mu(V)$ (respectively $\mu(W) \leqslant \mu(V)$). A non-zero sub-bundle $W \subset V$ with $\mu(W) \geqslant \mu(V)$ will be called a *destabilizing sub-bundle*.

2.3 Harder–Narasimhan filtration

Let V be a vector bundle on X. Then there exists a unique filtration (see [LeP97, § 5.4]), called the *Harder–Narasimhan filtration*, by sub-bundles

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{h-1} \subset V_h = V$$

such that V_i/V_{i-1} is semi-stable of slope μ_i and

$$\mu_1 > \mu_2 > \cdots > \mu_h.$$

The data of $(\dim(V_i/V_{i-1}), \mu_i)$ can be encoded into a polygon, called the *Harder–Narasimhan polygon* (see [LeP97, 11.1]). The Harder–Narasimhan polygon can be regarded as a measure of instability.

2.4 A measure of stability

Following [LN83], for a rank-2 vector bundle V, we put

$$s(V) = \deg(V) - 2\max\{\deg(L) : L \hookrightarrow V\},$$

where the maximum is taken over all rank-1 sub-bundles of V. By definition, $s(V) > 0$ (respectively $s(V) \geqslant 0$) if and only if V is stable (respectively semi-stable). When $s(V) \leqslant 0$, the information of $(s(V), \deg(V))$ is the same as that of the Harder–Narasimhan polygon of V. Therefore, one may regard s as a measure of stability extrapolating the Harder–Narasimhan polygons, although it is only for the rank-2 case (for possible variants for the higher rank case, see [BPL98]; for general reductive group, see [HN01]).

2.5 For any vector bundle with a connection (V, ∇), there exists a p-linear morphism of $\mathcal{O}_X$-modules, called the *p-curvature of* ∇,

$$\psi : T_X \to \operatorname{End}(V),$$

which measures the obstruction to the Lie algebra homomorphism $\nabla : T_X \to \operatorname{End}(V)$ being a homomorphism of p-Lie algebras. A connection is *p-flat* if ψ is zero. A vector bundle is *p-flat* if it admits a p-flat connection.

By a theorem of Cartier [Kat70, Theorem 5.1, p. 190], there exists a vector bundle W on X such that $F^*(W) \simeq V$ if and only if V carries a p-flat connection

$$\nabla : V \to \Omega^1_X \otimes V$$

such that the natural map

$$F^*(V^{\nabla=0}) \to V$$

(where $V^{\nabla=0}$ is the module of flat sections considered as an $\mathcal{O}_X$-module) is an isomorphism. If $V = F^*(W)$, then we write $\nabla^{\mathrm{Cartier}}$ for the canonical connection on W provided by Cartier's theorem.

2.6 Suppose (V, ∇) is a vector bundle with a connection and $W \subset V$ a sub-bundle. Then there is a natural map (the second fundamental form)

$$T_X \to \operatorname{Hom}(W, V/W),$$

which is zero if and only if ∇ preserves W. By Cartier's theorem, if (V, ∇) is p-flat and $W \hookrightarrow V$ is a sub-bundle preserved by ∇, then ∇ restricts to a p-flat connection on W.

2.7 Let B_1 be the vector bundle defined by the exact sequence

$$0 \to \mathcal{O}_X \to F_*(\mathcal{O}_X) \to B_1 \to 0.$$

The bundle B_1 is semi-stable of slope $g-1$ (and degree $(p-1)(g-1)$); moreover, for $p > 2$, $F^*(B_1)$ is not semi-stable [Ray82]. For $p = 2$, B_1 is a theta characteristic, i.e. $B_1^{\otimes 2} = \Omega_X^1$ [Ray82]. By [Ray78, Proposition 1.1], B_1 is stable when $p = 3$ and $g \geqslant 2$.

Lemma 2.8. *Let L be a line bundle on X. Then*

$$\det(F_* L) = \det(B_1) \otimes L.$$

Proof. See [Har77, ch. 4, Exercise 2.6]. ☐

2.9 Let V be a vector bundle on X. Then

$$\deg(F_* V) = \deg(V) + \operatorname{rank}(V) \deg(B_1).$$

This follows from Riemann–Roch and the fact that $\chi(F_* V) = \chi(V)$ or by § 2.8. In particular,

$$\mu(F_* V) = \frac{1}{p}\mu(V) + \left(1 - \frac{1}{p}\right)(g-1).$$

2.10 Duality

Let V be a vector bundle on X. Following [Ram87, § 1.16, p. 70], we have

$$F_*(V)^* \simeq F_*(V^* \otimes (\Omega_X^1)^{\otimes(1-p)}).$$

Thus the dual of $F_*(V)$ is of the form $F_*(V')$. We will often make use of this fact together with the following simple lemma.

Lemma. *Let V be a vector bundle of rank n on X, and m an integer such that $0 < m < n$. The following are equivalent:*

(i) *for all sub-bundles W of V of rank m, we have $\mu(W) < \mu(V)$ (respectively $\mu(W) \leqslant \mu(V)$);*

(ii) *for all sub-bundles W' of V^* of rank $n-m$, we have $\mu(W') < \mu(V^*)$ (respectively $\mu(W') \leqslant \mu(V^*)$).*

3. A general construction

Proposition 3.1. *Let V be a vector bundle on X. Then the adjunction map $F^*(F_*(V)) \to V$ is surjective and $\mu(F^*(F_*(V))) = \mu(V) + (p-1)(g-1) > \mu(V)$. In particular, $F^*(F_*(V))$ is unstable.*

Proof. The surjectivity of the adjunction map is easily check by a local calculation. The formula for the slope follows from § 2.9. Hence $\mu(F^*(F_*(V))) > \mu(V)$. ☐

Remark. In § 5.3, we prove a stronger assertion: $F^*(F_*(V))$ is highly unstable whenever V is semi-stable.

Proposition 3.2. *Let V be a semi-stable bundle on X.*

(i) *For any rank-1 sub-bundle L of F_*V, we have*

$$\mu(L) \leqslant \mu(F_*V) - \frac{(p-1)(g-1)}{p}.$$

(ii) *For any rank-2 sub-bundle E of F_*V, we have*

$$\mu(E) \leqslant \mu(F_*V) - \frac{1}{p}\left(\frac{pg}{2} - p - g + 1\right).$$

Proof. If $L \hookrightarrow F_*V$ is a line sub-bundle, then by adjunction there is a non-zero morphism $F^*L \to V$. Therefore, $\mu(F^*L) \leqslant \mu(V)$, i.e. $p \cdot \mu(L) \leqslant \mu(V) = p \cdot \mu(F_*V) - (p-1)(g-1)$. Here we have made use of the formulas in § 2.9. This proves (i).

Let $E \hookrightarrow F_*V$ be a sub-bundle of rank 2. Then by a theorem of Nagata (see [HN01]), there is a line sub-bundle $L \hookrightarrow E$ such that $\mu(L) \geqslant \mu(E) - g/2$. Thus we have $\mu(E) \leqslant \mu(L) + g/2 \leqslant \mu(F_*V) - (p-1)(g-1)/p + g/2$. This proves (ii). □

Theorem 3.3.

(i) *Let $p = 2$ and let V be a stable bundle of rank 2 and even degree on X. Then $F_*(V)$ is a semi-stable bundle of rank 4 and $F^*(F_*(V))$ is not semi-stable.*

(ii) *Suppose $p = 3$ (respectively $g \geqslant 3$ and $p = 5$). Let V be a line bundle on X. Then the bundle $F_*(V)$ is a stable bundle of rank 3 (respectively 5) and $F^*(F_*(V))$ is not semi-stable.*

Proof. (i) By § 2.10 and Proposition 3.1, it suffices to show that for any sub-bundle E of F_*V of rank $\leqslant 2$, we have $\mu(E) \leqslant \mu(F_*V)$. This is clear when $\operatorname{rank} E = 1$ by Proposition 3.2(i). Suppose that $\operatorname{rank} E = 2$ and $\mu(E) > \mu(F_*V)$. The proof of Proposition 3.2(ii) gives a line bundle $L \hookrightarrow E$ such that $\mu(E) \leqslant \mu(L) + g/2 \leqslant \mu(F_*V) + 1/2$.

The assumption that $\deg V$ is even implies that $\mu(F_*V) \in \frac{1}{2}\mathbb{Z}$. Thus we must have $\mu(E) = \mu(L) + g/2 = \mu(F_*V) + 1/2$. This gives $\mu(L) = \frac{1}{2}\mu(V)$ and $\mu(F^*L) = \mu(V)$, contradicting the stability of V as there is a non-zero morphism $F^*L \to V$ by adjunction.

(ii) By § 2.10 and Proposition 3.1, it suffices to check that F_*V does not have a destabilizing sub-bundle of rank $\leqslant 1$ (respectively $\leqslant 2$). This is immediate from Proposition 3.2. □

4. A detailed study of the case of rank 2 and characteristic two

Throughout this section, $p = 2$. We present our main results on the classification of rank-2 vector bundles destabilized by Frobenius, as well as the geometry of the Frobenius stratification.

4.1 A result on the Gunning bundle

We begin with an interesting observation about Gunning extensions, although this result is not needed in what follows. Recall that B_1 is a theta-characteristic [Ray82, § 4]. The unique non-trivial extension $0 \to B_1 \to W \to B_1^{-1} \to 0$ is called *the Gunning extension* and the bundle W is called *the Gunning bundle.*

Proposition. *Let ξ be a line bundle and $V = F_*(\xi \otimes B_1^{-1})$. The extension*

$$0 \to \xi \otimes B_1 \to F^*V \to \xi \otimes B_1^{-1} \to 0 \tag{$*$}$$

defines a class in $\operatorname{Ext}^1(\xi \otimes B_1^{-1}, \xi \otimes B_1) \simeq H^1(X, B_1^2) \simeq k$. *This class is trivial precisely when* $\deg(\xi \otimes B_1^{-1})$ *is even.*

Proof. Suppose that $\deg(\xi \otimes B_1^{-1})$ is even. Then we can write $L = \xi \otimes B_1^{-1} = M^2$. By [JX00, § 2], there is an exact sequence $0 \to M \to V \to M \otimes B_1 \to 0$. Pulling back by F, we get $0 \to L \to F^*V \to L \otimes B_1^2 \to 0$. This shows that $(*)$ splits.

Suppose that $L = \xi \otimes B_1^{-1}$ has odd degree $2n+1$. By a theorem of Nagata (see [LN83], cf. Remark in § 4.5), there is an exact sequence $0 \to M_1 \to V \to M_2 \to 0$, where M_1, M_2 are line bundles with degrees n and $n+g$, respectively. From the exact sequence $0 \to M_1^2 \to F^*V \to M_2^2 \to 0$, we deduce that $\dim \operatorname{Hom}(L, F^*V) \leqslant \dim \operatorname{Hom}(L, M_1^2) + \dim \operatorname{Hom}(L, M_2^2) = 0 + g = g$ by the Riemann–Roch formula. Since $\operatorname{Hom}(L, \xi \otimes B_1) = H^0(X, B_1^2)$ has dimension g, any morphism $L \to F^*V$ factors through the sub-module $\xi \otimes B_1$ in $(*)$. Therefore, $(*)$ does not split. □

Corollary. *Let W be the Gunning bundle and ξ a line bundle of degree $\equiv g \pmod 2$. Then there exists a stable bundle V such that $F^*V \simeq W \otimes \xi$. In particular, if g is even, then the Gunning bundle W is the Frobenius pull-back of a stable bundle.*

Remark. The proposition is implicit in [LS77] while, in the case of an ordinary curve with $g = 2$ and $p = 2$, the corollary is implicit in [LP02]. In [Gie73], Gieseker proved (by different methods) an analogous result in any characteristic when X is a Mumford curve.

4.2 The basic construction

Henceforth, fix an integer d. For an injection $V' \hookrightarrow V''$ of vector bundles of the same rank, define the *co-length* l of V' in V'' to be the length of the torsion $\mathcal{O}_X$-module V''/V'. Clearly, $s(V') \geqslant s(V'') - l$.

We now give a basic construction of stable vector bundles V of rank 2 with F^*V not semi-stable. Let $l \leqslant g-2$ be a non-negative integer, L a line bundle of degree $d-1-(g-2-l)$, and V a sub-module of F_*L of co-length l, then $\deg V = d$ and $s(V) \geqslant (g-1) - l > 0$ by Proposition 3.2. Therefore, V is stable.

On the other hand, by adjunction, there is a morphism $F^*V \to L$, and the kernel is a line bundle of degree $\geqslant d+1+(g-2-l) > d = \deg(F^*V)/2$. Therefore, F^*V is not semi-stable.

4.3 Exhaustion

Suppose that V is semi-stable of rank 2. Let $\xi = \det(V)$ and $d = \deg \xi = \deg V$. Let L, L' be line bundles with $\deg L$ being the smallest possible such that F^*V is an extension $0 \to L' \to F^*V \to L \to 0$. By adjunction, this provides a non-zero morphism $V \to F_*L$. Denote the image by M.

Suppose F^*V is not stable. Then $\deg L' \geqslant d$ and $\deg L \leqslant d$. Suppose $\operatorname{rank} M = 1$. Then $\deg M \geqslant d/2$ by semi-stability of V. By Proposition 3.2, $\deg M \leqslant (d+g-1)/2 - (g-1)/2 = d/2$. Hence $\deg M = d/2$. To conclude, either $\operatorname{rank} M = 2$ or $\deg M = d/2$.

Suppose F^*V is not semi-stable. Again by semi-stability of V and Proposition 3.2, $\operatorname{rank} M = 1$ would imply $\deg M \geqslant d/2$ and $\deg M \leqslant (d-1+g-1)/2 - (g-1)/2 = (d-1)/2$, a contradiction. Hence $\operatorname{rank} M = 2$. Since $\deg V = d$ and $\deg(F_*L) \leqslant d + (g-2)$, V is a sub-module of F_*L of co-length $l \leqslant g-2$, and $\deg L = d-1-(g-2-l)$. Thus, the basic construction yields all semi-stable vector bundles V of rank 2, with F^*V not semi-stable.

Corollary 4.3.1. *If V is semi-stable of rank 2 with F^*V not semi-stable, then V is actually stable.*

Corollary 4.3.2. *The basic construction with $l = g-2$ already yields all semi-stable vector bundles V of rank 2, with F^*V not semi-stable.*

Proof. In fact, if $l < l' \leqslant g-2$ and $L' = L \otimes \mathcal{O}(D)$ for some effective divisor D of degree $l'-l$ on X, then $V \hookrightarrow F_*L \hookrightarrow F_*L'$. Hence V is also a sub-module of F_*L' of co-length l'. Thus, V arises from the basic construction with (l', L') playing the role of (l, L). □

4.4 Classification

To ease the notation, set $d_l = d-1-(g-2-l)$. Let $\operatorname{Pic}^{d_l}(X)$ be the moduli space of line bundles of degree d_l on X, and let $\mathcal{L} \to \operatorname{Pic}^{d_l}(X) \times X$ be the universal line bundle.

Recall the convention in § 2.1. Let L be a line bundle and let $Q = Q_l = Q_{l,L} = \mathrm{Quot}_l(F_*L/X/k)$ be the scheme classifying sub-modules of F_*L of co-length l (see [Gro60/61, § 3.2]). Consider the $\mathbb{F}$-morphism $\varpi \times F : Q^{(p)} \times X \to Q \times X$ from § 2.1. Let

$$\mathcal{V} \hookrightarrow \mathcal{O}_Q \boxtimes F_*L = (\varpi \times F)_*(\mathcal{O}_Q \boxtimes L)$$

(sheaves on $Q \times X$) be the universal object on Q. By adjunction, we have a morphism $(\varpi \times F)^*\mathcal{V} \to \mathcal{O}_Q \boxtimes L$. Let $\mathcal{F}$ be the co-kernel. Then $\mathrm{pr}_*\mathcal{F}$ is a coherent sheaf on Q, where $\mathrm{pr} : Q \times X \to Q$ is the projection [Har77, II.5.20]. By [Har77, III.12.7.2], the subset

$$\{q \in Q : \dim_{\kappa(q)}((\mathrm{p}_*\mathcal{F}) \otimes \kappa(q)) > 0\}$$

is closed. Its complement is an open sub-scheme, denoted by $Q^* = Q^*_l = Q^*_{l,L}$, of Q. Then Q^* parameterizes those V with surjective $F^*V \to L$.

Let $\overline{\mathcal{M}}$ be the coarse moduli space of rank-2 S-equivalence classes of semi-stable vector bundles of degree d on X. Let $\mathcal{M}$ be the open sub-scheme parameterizing stable vector bundles and $\mathcal{M}_1(k) \subset \overline{\mathcal{M}}(k)$ the subset of those V such that F^*V is not semi-stable. By Corollary 4.3.1, $\mathcal{M}_1(k) \subset \mathcal{M}(k)$.

Proposition. *The basic construction gives a bijection*

$$\coprod_{\substack{0 \leqslant l \leqslant g-2 \\ \deg L = d-1-(g-2-l)}} Q^*_{l,L}(k) \to \mathcal{M}_1(k),$$

where the disjoint union is taken over all $l \in [0, g-2]$ and $L \in \mathrm{Pic}^{d_l}(X)$.

Proof. By § 4.3, the map is a surjection. Now suppose that $(l, L, V \subset F_*L)$ and $(l', L', V' \subset F_*L')$ give the same point in $\mathcal{M}_1(k)$, i.e. $V \simeq V'$. Since the unstable bundle F^*V has a unique quotient line bundle of degree $< \deg(V)/2$ (i.e. the second graded piece of the Harder–Narasimhan filtration), which is isomorphic to L, we must have $L = L'$. Consider the diagram

$$\begin{tikzcd}
F^*V \arrow[r] \arrow[d, "\wr"'] & L \arrow[d, equal] \\
F^*V' \arrow[r] & L'
\end{tikzcd}$$

where the vertical arrow is induced from an isomorphism $V \xrightarrow{\sim} V'$ and the horizontal arrows are the unique quotient maps. This diagram is commutative up to a multiplicative scalar in k^*. By adjunction, $V \hookrightarrow F_*L$ and $V' \hookrightarrow F_*L$ have the same image. In other words, $V = V'$ as sub-modules of F_*L. This proves the injectivity of the map. $\square$

4.5 Frobenius stratification

By [Gro60/61, 3.2], there is a scheme

$$\mathcal{Q} = \mathcal{Q}_l = \mathrm{Quot}_l((\varpi \times F)_*\mathcal{L}/(\mathrm{Pic}^{d_l}(X) \times X)/\mathrm{Pic}^{d_l} X) \xrightarrow{\pi} \mathrm{Pic}^{d_l} X$$

such that $\mathcal{Q}_x$ (the fiber at x) is $Q_{\mathcal{L}_x}$ for all $x \in (\mathrm{Pic}^{d_l} X)(k)$. By the same argument as before, there is an open sub-scheme $\mathcal{Q}^* \subset \mathcal{Q}$ such that $\mathcal{Q}^*_x = Q^*_{\mathcal{L}_x}$ for all $x \in \mathrm{Pic}^{d_l}(X)(k)$. The scheme $\mathcal{Q}$ is projective over $\mathrm{Pic}^{d_l}(X)$ (see [Gro60/61, 3.2]), and hence is proper over k. By checking the condition of formal smoothness (cf. [LeP97, 8.2.1]), it can be shown that $\mathcal{Q}$ is smooth over $\mathrm{Pic}^{d_l}(X)$, and hence is smooth over k.

The Frobenius stratification on the coarse moduli scheme $\overline{\mathcal{M}}$ is defined canonically using Harder–Narasimhan polygons of Frobenius pull-backs. Concretely, for $j \geqslant 0$, let P_j be the polygon from $(0,0)$ to $(1, d+j)$ to $(2, 2d)$. Let $\mathcal{M}_0 = \overline{\mathcal{M}}$, and for $j \geqslant 1$, let $\mathcal{M}_j(k)$ be the subset of $\overline{\mathcal{M}}(k)$

parameterizing those V such that the Harder–Narasimhan polygons [LeP97, 11.1] of F^*V lie above or are equal to P_j. Notice that $\mathcal{M}_1(k)$ agrees with the one defined in § 4.4.

As mentioned in the introduction, the existence of a universal bundle on $\mathcal{M}$ would imply that each $\mathcal{M}_j(k)$ is Zariski closed by Shatz's theorem [LeP97, 11.1, last remark]. In general, one can show that $\mathcal{M}_j(k)$ is closed by examining the geometric invariant theory construction of $\overline{\mathcal{M}}$. This fact also follows from our basic construction.

Theorem. *The subset $\mathcal{M}_j(k)$ is Zariski closed in $\overline{\mathcal{M}}(k)$, and hence underlies a reduced closed sub-scheme $\mathcal{M}_j$ of $\overline{\mathcal{M}}$. The scheme $\mathcal{M}_j$ is proper. The Frobenius stratum $\mathcal{M}_j \setminus \mathcal{M}_{j+1}$ is non-empty precisely when $0 \leqslant j \leqslant g-1$. For $1 \leqslant j \leqslant g-1$, write $l = g-1-j$. Then there is a canonical morphism*

$$\mathcal{Q}_l \to \overline{\mathcal{M}},$$

which has scheme-theoretic image $\mathcal{M}_j$ and induces a bijection from $\mathcal{Q}_l^*(k)$ to $\mathcal{M}_j(k) \setminus \mathcal{M}_{j+1}(k)$.

Proof. Suppose $0 \leqslant l \leqslant g-2$ and $j+l = g-1$. The universal object $\mathcal{V} \to \mathcal{Q}_l \times X$ is a family of stable vector bundles on X. This induces a canonical morphism $\mathcal{Q}_l \to \overline{\mathcal{M}}$. The image of $\mathcal{Q}_l(k)$ is precisely $\mathcal{M}_j(k)$ by (the proof of) Corollary 4.3.2. Since $\mathcal{Q}_l$ is proper, $\mathcal{M}_j$ is proper and closed in $\overline{\mathcal{M}}$. The rest of the proposition follows from §§ 4.4 and 4.3, and the fact that $\mathcal{Q}_l^*(k)$ is non-empty for $0 \leqslant l \leqslant g-2$ (see Lemma 4.6.3). □

Remark. By a theorem of Nagata (see [LN83, HN01]), $s(V) \leqslant g$ for all V. Therefore, $s(V) \leqslant g$ if $\deg V \equiv g \pmod 2$, and $s(V) \leqslant g-1$ if $\deg V \not\equiv g \pmod 2$. By Proposition 3.2, $V = F_*L$ achieves the maximum value of s among rank-2 vector bundles of the same degree. By the preceding theorem, vector bundles of the form $V = F_*L$ are precisely members of the smallest non-empty Frobenius stratum $\mathcal{M}_{g-1}$. Therefore, in a sense V is most stable yet F^*V is most unstable. More generally, for $1 \leqslant j \leqslant g-1$, we have (from § 4.2)

$$s(\mathcal{M}_j(k)) \geqslant \begin{cases} j & \text{if } d \equiv j \pmod 2, \\ j+1 & \text{if } d \not\equiv j \pmod 2. \end{cases}$$

Therefore, high instability of F^*V implies high stability of V.

4.6 Irreducibility

We will make use of the following simple lemma.

Lemma 4.6.1. *Let Y be a proper scheme over k, S an irreducible scheme of finite type over k of dimension s, r an integer $\geqslant 0$, and $f : Y \to S$ a surjective morphism. Suppose that all fibers of f are irreducible of dimension r. Then Y is irreducible of dimension $s + r$.*

Lemma 4.6.2. *The scheme $\mathcal{Q} = \mathcal{Q}_l$ is irreducible of dimension $2l + g$.*

Proof. There is a morphism [Gro60/61, § 6]

$$\delta : \mathcal{Q} \to \mathrm{Div}^l(X) = \mathrm{Sym}^l(X), \quad q \mapsto \sum_{P \in X(k)} \mathrm{length}_{\mathcal{O}_P}((F_*\mathcal{L}_{\pi(q)})/\mathcal{V}_q) \cdot P.$$

We claim that the morphism $\mathcal{Q} \to \mathrm{Div}^l(X) \times \mathrm{Pic}^{d_l}(X)$ is a surjection with irreducible fibers of dimension l. Indeed, the fiber at any $(D, L) \in \mathrm{Div}^l(X)(k) \times \mathrm{Pic}^{d_l}(X)(k)$ is the same as the fiber at D of the analogous morphism $Q_{l,L} \to \mathrm{Div}^l(X)$. If $D = \sum n_i x_i$ ($x_i \neq x_j$ for $i \neq j$, $\sum n_i = l$), to give a point on the fiber is to give a length-n_i quotient of $(F_*L)_{x_i}$, for each i. By (the proof of) Lemma 5.2 of [MX02], for fixed i, the space of such quotients is non-empty and irreducible of dimension $n_i \cdot (\dim F_*L - 1) = n_i$. Therefore, the fiber is irreducible of dimension $\sum n_i = l$. Now the result follows from Lemma 4.6.1 since $\mathcal{Q}$ is proper. □

Lemma 4.6.3. *$\mathcal{Q}^*$ is open and dense in $\mathcal{Q}$.*

Proof. By the construction in §§ 4.4 and 4.5, $\mathcal{Q}^*$ is open in $\mathcal{Q}$. Since $\mathcal{Q}$ is irreducible of dimension $2l+g$, it suffices to show that $\mathcal{Q}^*$ is non-empty. In fact, $\mathcal{Q}^*$ contains an open subset of dimension $2l+g$ as shown below.

Let $B(X,l) \subset \mathrm{Div}^l(X)$ be the open sub-scheme parameterizing multiplicity-free divisors of degree l, also known as the configuration space of unordered l points in X. Let U be the inverse image of $B(X,l) \times \mathrm{Pic}^{d_l}(X)$ under $\mathcal{Q} \to \mathrm{Div}^l(X) \times \mathrm{Pic}^{d_l}(X)$. According to the proof of the preceding lemma, each fiber of $U \to B(X,l) \times \mathrm{Pic}^{d_l}(X)$ is isomorphic to $(\mathbb{P}^1)^l$ (this is the case of $n_i = 1$ for each i; so we get the product of l copies of $\mathbb{P}^1$, which is the space of length 1 quotients of $(F_*L)_x \simeq k[[\pi_x]]^2$). Let $U^* = U \cap \mathcal{Q}^*$. Then each fiber of $U^* \to B(X,l) \times \mathrm{Pic}^{d_l}(X)$ is isomorphic to W^l, where $W \subset \mathbb{P}^1$ is the space described below. Let $A = B = k[[\pi_x]]$, let $f : A \to B$ be the map $a \mapsto a^2$, and let ${}_AB$ denote B as an A-module via f. Then $\mathbb{P}^1$ (respectively W) is the space of A-submodules $V \subset {}_AB$ such that ${}_AB/V \simeq A/\pi_x A$ (respectively and in addition, $V \otimes_f B \to B$ is surjective). It is clear that $\mathbb{P}^1 \setminus W$ is a single point corresponding to $V = {}_A(\pi_x B)$. Therefore, $W \simeq \mathbb{A}^1$ and U^* is an open subset of $\mathcal{Q}^*$ of dimension $2l+g$. □

Theorem 4.6.4. *For $1 \leqslant j \leqslant g-1$, $\mathcal{M}_j$ is proper, irreducible, and of dimension $g + 2(g-1-j)$. In particular, $\mathcal{M}_1$ is irreducible and of dimension $3g-4$.*

4.7 Fixing the determinant

Fix a line bundle ξ of degree d. Let $\overline{\mathcal{M}}(\xi) \subset \overline{\mathcal{M}}$ be the closed sub-scheme of $\overline{\mathcal{M}}$ parameterizing those V with $\det(V) = \xi$. Let $\mathcal{M}_j(\xi) = \overline{\mathcal{M}}(\xi) \cap \mathcal{M}_j$ for $j \geqslant 0$.

Remark. For $1 \leqslant j \leqslant g-1$, $\dim \mathcal{M}_j(\xi) = 2(g-1-j)$. In particular, $\dim \mathcal{M}_1(\xi) = 2(g-2)$.

Proof. Since $\mathcal{M}_j(\xi)$ is nothing but the fiber of the surjective morphism $\det : \mathcal{M}_j \to \mathrm{Pic}^d(X)$, it has dimension $2(g-1-j)$ for a dense open set of $\xi \in \mathrm{Pic}^d(X)(k)$. However, $\mathcal{M}_j(\xi_1)$ is isomorphic to $\mathcal{M}_j(\xi_2)$ for all $\xi_1, \xi_2 \in \mathrm{Pic}^d(X)(k)$, via $V \mapsto V \otimes L$, where $L^2 \simeq \xi_2 \otimes \xi_1^{-1}$. Thus the remark is clear. □

A slight variation of the above argument shows that $\mathcal{M}_j(\xi)$ is irreducible. Alternatively, assume $1 \leqslant j \leqslant g-1$. Let $l = g-1-j$ and let $\mathcal{Q}(\xi) = \mathcal{Q}_l(\xi)$ be the inverse image of ξ under $\mathcal{Q} \to \mathrm{Pic}^d(X)$, $q \mapsto \det(\mathcal{V}_q)$. Since $\det(\mathcal{V}_q) = B_1 \otimes \mathcal{L}_{\pi(q)} \otimes \mathcal{O}(-\delta(q))$, the morphism $\det : \mathcal{Q} \to \mathrm{Pic}^d(X)$ factors as

$$\mathcal{Q} \to \mathrm{Div}^l(X) \times \mathrm{Pic}^{d_l}(X) \xrightarrow{\psi} \mathrm{Pic}^d(X),$$

where ψ is $(D, L) \mapsto B_1 \otimes L \otimes \mathcal{O}(-D)$. It is clear that $\psi^{-1}(\xi)$ is isomorphic to $\mathrm{Div}^l(X)$, and hence is an irreducible variety.

The fibers of $\mathcal{Q}(\xi) \to \psi^{-1}(\xi)$ are just some fibers of $\mathcal{Q} \to \mathrm{Div}^l(X) \times \mathrm{Pic}^{d_l}(X)$; hence they are irreducible of dimension l as in the proof of Lemma 4.6.2. Being a closed sub-scheme of $\mathcal{Q}$, $\mathcal{Q}(\xi)$ is proper, thus and irreducible by Lemma 4.6.1. Now it is easy to deduce the following.

Theorem. *There is a canonical (Frobenius) stratification by Harder–Narasimhan polygons*

$$\emptyset = \mathcal{M}_g(\xi) \subset \mathcal{M}_{g-1}(\xi) \subset \cdots \subset \mathcal{M}_0(\xi) = \overline{\mathcal{M}}(\xi),$$

with $\mathcal{M}_j(\xi)$ non-empty, proper, irreducible, and of dimension $2(g-1-j)$ for $1 \leqslant j \leqslant g-1$.

4.8 A variant

Let $\mathcal{M}'(k)$ be the subset of $\overline{\mathcal{M}}(k)$ consisting of the S-equivalence classes of those V such that F^*V are not stable. Clearly, $\mathcal{M}'(k) \supset \mathcal{M}_1(k)$.

By Corollary 4.3.1, the closed subset $\mathcal{M}^{\mathrm{ns}}(k) = \overline{\mathcal{M}}(k) \setminus \mathcal{M}(k)$ is contained in $\mathcal{M}'(k) \setminus \mathcal{M}_1(k)$. By § 4.3, if $V \in \mathcal{M}'(k) \setminus \mathcal{M}^{\mathrm{ns}}(k)$, there is a line bundle L of degree d such that $V \hookrightarrow F_*L$ is a sub-module of co-length $\leqslant g-1$. Conversely, the argument of § 4.2 shows that if V is of co-length $\leqslant g-1$ in F_*L for some L of degree d, then $V \in \mathcal{M}'(k)$.

Thus we conclude that $\mathcal{M}'(k)$ is the union of $\mathcal{M}^{\mathrm{ns}}(k)$ and the image $\mathcal{M}'_0(k)$ of $\mathcal{Q}_{g-1}(k)$ for a suitable morphism $\mathcal{Q}_{g-1} \to \overline{\mathcal{M}}$, where $\mathcal{Q}_{g-1}$ is defined in § 4.5. It follows that $\mathcal{M}'_0(k)$ and $\mathcal{M}'(k)$ are Zariski closed in $\overline{\mathcal{M}}(k)$, and hence are sets of k-points of reduced closed sub-schemes $\mathcal{M}'_0$ and $\mathcal{M}'$ of $\overline{\mathcal{M}}$.

THEOREM. *The scheme $\mathcal{M}'_0$ is irreducible of dimension $3g-2$. It contains two disjoint closed subsets: $\mathcal{M}'_0 \cap \mathcal{M}^{\mathrm{ns}}$, which is irreducible of dimension $2g-1$ when d is even and empty when d is odd; and $\mathcal{M}_1$, which is irreducible of dimension $3g-4$.*

Proof. Since $\mathcal{Q}_{g-1}$ is irreducible, $\mathcal{M}'_0$ is irreducible. We now analyze $\mathcal{M}'_0 \cap \mathcal{M}^{\mathrm{ns}}$. Suppose that $V \in \mathcal{Q}_{g-1}(k)$ and is a representative of an S-equivalence class in $\mathcal{M}^{\mathrm{ns}}(k)$. Then $d = \deg V$ is even and there exists L of degree d such that V is a sub-module of F_*L of co-length $g-1$. By assumption, there exists an exact sequence $0 \to M \to V \to N \to 0$ with $\deg M = \deg N = d/2$. Adjunction applied to the composition $M \hookrightarrow V \hookrightarrow F_*L$ provides a non-zero morphism $F^*M = M^2 \to L$. This implies that $F^*M \simeq L$. It follows that the morphism $F^*V \to L$ is onto and therefore $F^*V \simeq M^2 \oplus N^2$.

By adjunction, there is only one non-zero morphism $M \to F_*(F^*M)$ (modulo k^*). By [JX00, § 2], the morphism $M \to F_*(F^*M)$ is part of an exact sequence $0 \to M \to F_*(F^*M) \to M \otimes B_1 \to 0$. Starting with a sub-module of $M \otimes B_1$ of co-length $g-1$, we obtain a vector bundle $V \in \mathcal{M}'_0(k) \cap \mathcal{M}^{\mathrm{ns}}(k)$ as the inverse image of that sub-module in $F_*(F^*M)$. In other words, V fits into the following commutative diagram in which the square on the right is Cartesian.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "id"] & V \arrow[r] \arrow[d, hook] & M' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & F_*(F^*M) \arrow[r] & M \otimes B_1 \arrow[r] & 0
\end{tikzcd}$$

The sub-modules of $M \otimes B_1$ of co-length $g-1$ are of the form $M \otimes B_1 \otimes \mathcal{O}(-D)$ for $D \in \mathrm{Div}^{g-1}(X)(k)$. Thus there is a morphism $\pi' : \mathcal{Q}' = \mathrm{Div}^{g-1}(X) \times \mathrm{Pic}^{d/2}(X) \to \overline{\mathcal{M}}$ inducing a surjection $\mathcal{Q}'(k) \to \mathcal{M}'_0(k) \cap \mathcal{M}^{\mathrm{ns}}(k)$. We claim that this morphism is generically finite of separable degree 2. This claim implies that $\mathcal{M}'_0 \cap \mathcal{M}^{\mathrm{ns}}$ is irreducible of dimension $2g-1$.

Indeed, there is a dense open subset U of $\mathrm{Div}^{g-1}(X)$ such that $D \in U(k)$ if and only if $h^0(\mathcal{O}(D)) = 1$ [Har77, III.12.8]. If $(D, M) \in (U \times \mathrm{Pic}^{d/2}(X))(k)$, then $h^0(\Omega^1_X \otimes \mathcal{O}(-D)) = h^0(\mathcal{O}(D))$ by Riemann–Roch. Hence, $\Omega^1_X \otimes \mathcal{O}(-D) = \mathcal{O}(D')$ for a unique $D' \in U(k)$. This defines an involution θ on $U \times \mathrm{Pic}^{d/2}(X)$ by $\theta(D, M) = (D', M')$ with $M' = M \otimes B_1 \otimes \mathcal{O}(-D)$. We now show that if $(D, M) \in U(k)$ and $\pi'(D, M)$ is the S-equivalence class $[V]$ of V, then $\pi^{-1}([V]) = \{(D, M), (D', M') := \theta(D, M)\}$. Indeed, if $\pi(M_1, D_1) = [V]$, then M_1 is one of the two Jordan–Hölder factors of $[V]$, namely M and M'. Knowing M_1, one can determine D_1 uniquely by the condition $0 \to M_1 \to V \to M_1 \otimes B_1 \otimes \mathcal{O}(-D_1) \to 0$. The fixed-point set of θ is $U^\theta = \{(D, M) : \mathcal{O}(D) \simeq B_1\}$, which is a proper closed subset. Then $\pi'|(U \setminus U^\theta)$ is two-to-one. This proves the claim.

Next, we consider the morphism $\mathcal{Q}_{g-1} \to \mathcal{M}'_0$. It induces a surjection $\mathcal{Q}^*_{g-1}(k) \twoheadrightarrow \mathcal{M}'_0(k) \setminus \mathcal{M}_1(k)$. Again the claim is that the morphism is generically finite of separable degree at most 2 over the open set $\mathcal{M}'_0(k) \setminus (\mathcal{M}_1(k) \cup \mathcal{M}^{\mathrm{ns}}(k))$. This claim implies that $\mathcal{M}'_0$ is irreducible of dimension $3g-2$.

Suppose $q \in \mathcal{Q}^*_{g-1}(k)$ and q gives rise to a stable $V \in \mathcal{M}'_0(k)$. Notice that the S-equivalence class of V is just $\{V\}$. There is an exact sequence $0 \to L \otimes B_1^2 \otimes \mathcal{O}(-2\delta(q)) \to F^*V \to L \to 0$, where $L = \mathcal{L}_{\pi(q)}$. By semi-stability, F^*V has at most two quotient line bundles of degree d, say $F^*V \to L_1$

and $F^*V \to L_2$. Then q must be one of the two data $V \hookrightarrow F_*L_1$ or $V \hookrightarrow F_*L_2$ provided by adjunction. This proves the claim. $\square$

Remark. (i) Assume that d is even and $V \in \mathcal{Q}_{g-1}(k)$ represents an S-equivalence class in $\mathcal{M}^{\mathrm{ns}}(k)$. The above proof shows that F^*V always splits as $M^2 \oplus M'^2$. However, V seldom splits into the direct sum of two line bundles. Indeed, $V \simeq M \oplus M'$ implies $M^2 \simeq L \simeq M'^2$.

The locus of those $V \in \mathcal{M}^{\mathrm{ns}}(k)$ such that the two simple factors M and M' of V satisfying $M^2 \simeq M'^2$ is closed of dimension g. Therefore, for most points x in $\mathcal{M}'_0(k) \cap \mathcal{M}^{\mathrm{ns}}(k)$, which is of dimension $2g-1 > g$, when we represent the S-equivalence class x by $V \in \mathcal{Q}_{g-1}(k)$, V does not split into the sum of two line bundles. In particular, $\mathcal{Q}_{g-1}(k)$ may contain a vector bundle V without containing the whole S-equivalence class of V.

(ii) The stratum $\mathcal{M}' \setminus \mathcal{M}_1$ is the first in the s-stratification [LN83] which is not a Frobenius stratum. The other s-strata are more complicated and not pursued here.

4.9 Example

When $g = 2$, $\mathcal{M}_1(\xi)$ is a single point, corresponding to the vector bundle $F_*(\xi \otimes B_1^{-1})$. When $\xi = B_1$, this refines a result of [JX00, 1.1], which says that $\mathcal{M}_1(\xi)$ is a single $\mathrm{Pic}(X)[2]$-orbit.

When $\xi = \mathcal{O}$, again $\mathcal{M}_1(\xi)$ consists of a single point [LP02]. Our result also extends a theorem of Mehta (see [JX00, 3.2]), which states that there are only finitely many rank-2 semi-stable vector bundles V on X with $\det(V) = \mathcal{O}$ and F^*V not semi-stable when $p \geqslant 3, g = 2$. We now have this result for $p = 2, g = 2$ with the stronger conclusion of uniqueness.

5. Pre-opers and opers

This section is largely inspired by the work of Beilinson and Drinfel'd [BD00]. We show that pre-opers with connections of p-curvature zero provide, under additional assumptions, examples of Frobenius destabilized bundles. In small characteristics we describe the lowest Frobenius stratum in terms of pre-opers.

5.1 Pre-opers

Let V be a vector bundle on X with a flat connection ∇. Suppose that $\{V_i\}_{0 \leqslant i \leqslant l} \subset V$ is an increasing filtration by sub-bundles such that:

(i) $V_0 = 0$, $V_l = V$;

(ii) $\nabla(V_i) \subset V_{i+1} \otimes \Omega^1_X$ for $0 \leqslant i \leqslant l-1$;

(iii) $V_i/V_{i-1} \xrightarrow{\nabla} (V_{i+1}/V_i) \otimes \Omega^1_X$ is an isomorphism for $1 \leqslant i \leqslant l-1$;

Then $(V, \nabla, \{V_i\})$ is said to be a *pre-oper*. A pre-oper is *p-flat* if ∇ has p-curvature zero.

Remark. (i) Let $(V, \nabla, \{V_i\}_{0 \leqslant i \leqslant l})$ be a pre-oper. If $g \geqslant 2$ and V_1/V_0 is semi-stable, then the filtration $\{V_i\}_{0 \leqslant i \leqslant l}$ is nothing but the Harder–Narasimhan filtration of V.

(ii) A direct computation shows that $\mu(V_1) = \mu(V) + (l-1)(g-1)/r_1$, where $r_1 = \mathrm{rank}(V_1) = \mathrm{rank}(V_i/V_{i-1})$.

5.2 Opers

Let $\mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X)$ be the ring of differential operators on X. An *oper* $(V, \nabla, \{V_i\}_{0 \leqslant i \leqslant l})$ is a pre-oper such that the connection ∇ on V extends to a structure of $\mathcal{D}_X$-module on V.

Remark. (i) By a Theorem of Katz (see [Gie75, Theorem 1.3, p. 4]), a locally free $\mathcal{O}_X$-module V admits the structure of a $\mathcal{D}_X$-module if and only if there exists a sequence of vector bundles $\{V^i\}_{i \geqslant 0}$

such that $V^0 = V$ and $F^*(V^{i+1}) = V^i$ for $i \geqslant 0$. Moreover, if (V, ∇) is a vector bundle such that ∇ extends to a $\mathcal{D}_X$-module structure on V, then $V^1 = V^{\nabla=0}$ and the p-curvature of ∇ is zero. Therefore, an oper is p-flat.

(ii) When $l = \operatorname{rank} V$, what we have called an oper here is the same as an GL_l-oper as defined in [BD00].

(iii) A p-flat oper with underlying bundle of rank 2 is the same as a dormant torally indigenous bundle in [Moc99].

5.3 A canonical filtration

Let W be a vector bundle on X. We define a canonical increasing filtration on $V = F^*(F_*W)$ by abelian sub-sheaves $\{V_i\}_{0\leqslant i\leqslant p}$ as follows:

$$\begin{aligned} V_p &= V, \\ V_{p-1} &= \ker(V_p = F^*(F_*W) \to W), \\ V_i &= \ker(V_{i+1} \xrightarrow{\nabla^{\mathrm{Cartier}}} V \otimes \Omega^1_X \to (V/V_{i+1}) \otimes \Omega^1_X), \quad 0 \leqslant i \leqslant p-2. \end{aligned}$$

It is elementary to check by induction that each V_i is actually an $\mathcal{O}_X$-sub-module of V.

Theorem.

(i) *V_p/V_{p-1} is isomorphic to W.*
(ii) *$(V, \nabla^{\mathrm{Cartier}}, \{V_i\}_{0\leqslant i\leqslant p})$ is a pre-oper.*
(iii) *If $g \geqslant 2$ and W is semi-stable, $\{V_i\}_{0\leqslant i\leqslant p}$ is simply the Harder–Narasimhan filtration on V.*

Proof. By Remark 5.1(i), statements (i) and (ii) of the Theorem imply (iii). Statement (i) of the Theorem is simply Proposition 3.1. To prove (ii) of the Theorem, we note that the definition of pre-opers and the formation of the filtration $\{V_i\}_{0\leqslant i\leqslant p}$ can be made on any smooth one-dimensional noetherian scheme over k, and statements (i) and (ii) of the Theorem make sense in this context. In fact, the statements being local, we are reduced to the case of a free $\mathcal{O}_X$-module W. Moreover, all the relevant formations commute with direct sums, and hence we are reduced to the case of $W = \mathcal{O}_X$.

We can even reduce to the case $X = \operatorname{Spec} k[[t]]$ and use an explicit calculation to complete the proof. Alternatively, one can check that the construction of [Ray82, Remarques 4.1.2(2)] gives the same filtration and proves the theorem. ☐

Proposition. *Assume that F_*L is stable for any line bundle L. Let $\left(V, \nabla, \{V_i\}_{0\leqslant i\leqslant p}\right)$ be a p-flat pre-oper with $\operatorname{rank}(V) = p$ and $V^{\nabla=0}$ stable. Then $V^{\nabla=0} \simeq F_*L$ for a suitable line bundle L on X.*

Proof. As $0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = V$ is a pre-oper of rank p, $V/V_{p-1} = L$ is a line bundle. The morphism $F^*(V^{\nabla=0}) = V \to V/V_{p-1} = L$ gives by adjunction a non-zero morphism $V^{\nabla=0} \to F_*L$. Since these bundles are stable and of the same degree, the map $V^{\nabla=0} \to F_*L$ is an isomorphism. ☐

Remark. Let L be a line bundle and $g \geqslant 2$.

(i) The assumption that F_*L is stable is no longer needed. After this paper was written, Lange and Pauly [LP03, § 1] showed that F_*L is always stable. Thus any p-flat pre-oper V of rank p with $V^{\nabla=0}$ stable is of the form $(V = F^*(F_*(L)), \nabla^{\mathrm{Cartier}})$ together with the Harder–Narasimhan filtration. Suppose this pre-oper is an oper. Then $F^*(V)$ is a $\mathcal{D}_X$-module with infinite Frobenius descent by Remark 5.2(i). This implies $\deg(F_*L) = \deg(L) + (p-1)(g-1) = 0$, otherwise, this pre-oper is not an oper.

(ii) Theorem 5.3 shows that $V = F^*(F_*(L))$ is highly unstable and it is likely that V is in a minimal Frobenius stratum (this is indeed the case in characteristic two). At least, the bound in [Sun99, Theorem 3.1, p. 51] is reached by V when the rank is p.

5.4 The underlying bundle of a pre-oper is typically unstable. In some circumstances, the Frobenius descent of a p-flat pre-oper is (semi)-stable. This provides a way of constructing Frobenius destabilized bundles in terms of pre-opers.

Proposition. *Let $(V, \nabla, \{V_i\}_{0\leqslant i\leqslant l})$ be a p-flat pre-oper with V_1 semi-stable of rank r_1.*

(i) *Suppose $p > l(l-1)(lr_1-1)(g-1)$. Then $V^{\nabla=0}$ is semi-stable.*

(ii) *Suppose $r_1 = 1$ and $p > l(l-1)^2(g-1)$. Then $V^{\nabla=0}$ is stable.*

Proof. (i) Recall from Remark 5.1(ii) that $\mu(V_1) = \mu(V) + (l-1)(g-1)/r_1$. Since V_1 is semi-stable, $\{V_i\}_{0\leqslant i\leqslant l}$ is the Harder–Narasimhan filtration of V. In particular, if $W \subset V$, then $\mu(W) \leqslant \mu(V_1)$.

Let $V' = V^{\nabla=0}$. Then $F^*(V') = V$. Suppose that V' is not semi-stable, and $W' \subset V'$ is such that $\mu(W') > \mu(V')$. Then $W = F^*(W')$ satisfies $\mu(W) \leqslant \mu(V_1)$, and

$$\frac{\mu(V)}{p} = \mu(V') < \mu(W') = \frac{\mu(W)}{p} \leqslant \frac{\mu(V_1)}{p} = \frac{\mu(V)}{p} + \frac{(l-1)(g-1)}{p \cdot r_1}.$$

However, $\mu(V')$ (respectively $\mu(W')$) is a fraction of the form a/b, with $a, b \in \mathbb{Z}$, $0 < b \leqslant l \cdot r_1$ (respectively $0 < b < l \cdot r_1$). Therefore, $\mu(W') - \mu(V') \geqslant 1/(l \cdot r_1 \cdot (l \cdot r_1 - 1))$. This contradicts the assumption on p.

(ii) Let $V' = V^{\nabla=0}$ and $0 = W'_0 \subset \cdots \subset W'_s = V'$ be a Jordan–Hölder series for V'. Then each W'_i/W'_{i-1} is stable of slope μ/p, where $\mu = \mu(V)$. Let $W_i = F^*(W'_i/W'_{i-1})$, $\mu_{\max}(W_i)$ be the largest possible slope of sub-bundles of W_i, and $\mu_{\min}(W_i)$ be the smallest possible slope of quotient bundles of W_i. By definition, $\mu_{\min}(W_i) \leqslant \mu \leqslant \mu_{\max}(W_i)$.

Let i_0 be the smallest integer such that $F^*(W'_{i_0}) \to V_l/V_{l-1}$ is non-zero. Then $\mu_{\min}(W_{i_0}) \leqslant \mu(V_l/V_{l-1}) = \mu - (l-1)(g-1)$. Similarly, there exists an index i_1 such that $\mu_{\max}(W_{i_1}) \geqslant \mu(V_1) = \mu + (l-1)(g-1)$.

A theorem of Sun [Sun99, Theorem 3.1] asserts that

$$\mu_{\max}(W_i) - \mu_{\min}(W_i) \leqslant (\operatorname{rank}(W_i) - 1)(2g-2).$$

This implies that $\operatorname{rank}(W_{i_0}) \geqslant (l+1)/2$ and $\operatorname{rank}(W_{i_1}) \geqslant (l+1)/2$. Thus, $i_0 = i_1$ and

$$(l-1)(2g-2) \leqslant \mu_{\max}(W_{i_0}) - \mu_{\min}(W_{i_0}) \leqslant (\operatorname{rank}(W_{i_0}) - 1)(2g-2)$$

by Sun's theorem again. Therefore, $\operatorname{rank} W_{i_0} \geqslant l$ and this forces W'_{i_0} to be the only Jordan–Hölder factor of V'. □

Remark. The bound on p can often be improved for particular $(l, g, r_1, \deg(V_1))$. This is clear from the proof.

Acknowledgements

The results of this paper were discovered independently by the first two authors and the last two authors. When it was realized that there was a considerable overlap in the techniques and the results, it was decided to write it jointly. The last two named authors thank C.-L. Chai for discussions and also wish to thank the hospitality of the National Center for Theoretical Sciences, Hsinchu, Taiwan. Finally, we thank the referee for a very careful reading of the manuscript and for their insightful comments and suggestions for improvement.

References

AB82 M. F. Atiyah and R. Bott, *The Yang–Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London. Ser. A **308** (1982), 523–615.

BD00 A. Beilinson and V. Drinfel'd, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Preprint (2000).

BPL98 L. Brambila Paz and H. Lange, *A stratification of the moduli space of vector bundles on curves*, J. reine angew. Math. **494** (1998), 173–187.

Cha00 C.-L. Chai, *Newton polygons as lattice points*, Amer. J. Math. **122** (2000), 967–990.

Gie73 D. Gieseker, *Stable vector bundles and the Frobenius morphism*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 95–101.

Gie75 D. Gieseker, *Flat vector bundles and the fundamental group in non-zero characteristics*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), 1–31.

Gro60/61 A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert* (Seminaire Bourbaki, tome 13, 1960/61, no. 221), in *Fondements de la géométrie algébrique* (Secrétariat Math., Paris, 1962).

Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, Heidelberg, 1977).

HN01 Y. I. Holla and M. S. Narasimhan, *A generalisation of Nagata's theorem on ruled surfaces*, Compositio Math. **127** (2001), 321–332.

Jos03 K. Joshi, *Frobenius destabilized bundles III*, Preprint (2003).

JX00 K. Joshi and E. Z. Xia, *Moduli of vector bundles on curves in positive characteristics*, Compositio Math. **122** (2000), 315–321.

Kat70 N. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Publ. Math. Inst. Hautes Études Sci. **39** (1970), 175–232.

LN83 H. Lange and M. S. Narasimhan, *Maximal sub-bundles of rank two vector bundles on curves*, Math. Ann. **266** (1983), 55–72.

LP03 H. Lange and C. Pauly, *On Frobenius-destabilized rank-*2 *vector bundles on curves*, Preprint (2003), arXiv:math.AG/0309456.

LS77 H. Lange and U. Stuhler, *Vectorbündel auf Kurven und Darstellungen der Algebraischen Fundamentalgruppe*, Math. Z. **156** (1977), 73–83.

LP02 Y. Laszlo and C. Pauly, *The action of the Frobenius map on rank* 2 *vector bundles in characteristic* 2, J. Algebraic Geom. **11** (2002), 219–243.

LP04 Y. Laszlo and C. Pauly, *Frobenius map, vector bundles and Kummer's quartic in characteristic* 2 *and* 3, Adv. Math. **185** (2004), 246–269.

LeP97 J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Advanced Mathematics, vol. 54 (Cambridge University Press, 1997).

MX02 E. Markman and E. Z. Xia, *The moduli of flat* $\mathrm{PU}(p,p)$*-structures with large Toledo invariants*, Math. Z. **204** (2002), 95–109.

Moc99 S. Mochizuki, *Foundations of p-adic Teichmüller theory*, AMS/IP Studies in Advanced Mathematics, vol. II (American Mathematical Society, Providence, RI, 1999).

Oss04 B. Osserman, *Limit linear series in positive characteristic and Frobenius-unstable vector bundles on curves*, PhD dissertation, Massachusetts Institute of Technology, 2004.

RR84 S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), 269–291.

Ram87 A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Publ. Math. Inst. Hautes Études Sci. **65** (1987), 61–90.

Ray78 M. Raynaud, *Contre-exemple au "vanishing theorem" en caractéristique* $p > 0$, in *C. P. Ramanujam – a tribute*, Tata Inst. Fundam. Res. Stud. Math., vol. 8 (Springer, Berlin, 1978), 273–278.

Ray82 M. Raynaud, *Sections des fibrés vectoriels sur une courbe*, Bull. Soc. Math. France **110** (1982), 103–125.

Sun99 X. Sun, *Remarks on semistability of G-bundles in positive characteristic*, Compositio Math. **119** (1999), 41–52.

Kirti Joshi kirti@math.arizona.edu
Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

S. Ramanan ramanan@math.tifr.res.in
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

Eugene Z. Xia ezxia@mail.ncku.edu.tw
Department of Mathematics, National Cheng Kung University, Tainan 701, Taiwan, Republic of China

Jiu-Kang Yu jyu@math.purdue.edu
Department of Mathematics, Purdue University, West Lafayette, IN 47906, USA